Journal of Engineering Mathematics 17 (1983) 149–189
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Some static problems for the nonlinear elastic string

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(Received October 8, 1982)

Summary

Finite elastic deformations of an elastic string subjected to a vertical and a normal force are investigated, supplementing earlier results by Dickey and adding some new results, including considerations of the stability of the solutions. The relation of the solutions of the exact nonlinear theory to an approximate engineering theory of Föppl is discussed.

1. Introduction

In this paper we consider the problem of a nonlinear elastic string, discussed earlier by Dickey [1,2] and Carrier [3]. In particular, reference is made to Chapter 2 of [1], where the problem to describe all possible equilibrium solutions of a string acted upon by a constant vertical force is posed. We here discuss some extensions and additional results. Furthermore, it is felt that the description given by Dickey deserves some comments.

We first rectify an error in the solution for the inextensible string. The elementary approximate theory of Föppl is then given a more complete treatment including a discussion of the stability of the solutions obtained in [1], and the relation of the solution of the exact fully nonlinear theory of elastic strings to the Föppl solution is considered. The range of validity of the Föppl approximation has previously been determined by the second-named author for nonlinear circular elastic membrane problems [4]. The instability of the compressive exact solutions of [1] is obtained on the basis of the energy criterion.

Next we obtain a new solution for the normally loaded string, which is multi-valued in certain ranges of the load parameter, in some way analogous to the multiplicity of compressive solutions of a vertically loaded circular membrane [5], but different from the vertically loaded string. It is found that multiple tensile solutions may also exist, in contrast to the vertically loaded string, where the tensile solution for a given load is always unique. If the load is conservative, compressive solutions are again shown to be unstable.

Finally the stability of the static solutions is examined by means of the kinetic method, which provides a generally more acceptable stability criterion than the static potential energy criterion. The latter is restricted to conservative problems; moreover, its applicability in continuum mechanics has not been rigorously established as yet (e.g. see the discussion by Koiter [6]).

* Part of the results were presented by the second-named author at a meeting on Continuum Mechanics in Oberwolfach, January 1980.

2. The inextensible string

Assuming symmetric deformations, with $2L$ the length of the string and $2l$ the distance of its end points, the equilibrium position of a string subjected to a constant vertical force is given by the variational problem [1]

$$V=\int_0^l y\sqrt{1+(y')^2}\,\mathrm{d}x=\text{stationary},\qquad \text{subject to } \int_0^l \sqrt{1+(y')^2}\,\mathrm{d}x=L \tag{1}$$

where $y(x)$ must satisfy $y'(0)=0$ and $y(l)=a>0$. The solution of the Euler equation of (1) is the well-known catenary

$$y(x)=A\cosh(x/A)\text{-}\lambda.$$

The constant of integration A is determined from the constraint in (1), which leads to the condition

$$\sinh(l/A)=L/A=(L/l)/(l/A). \tag{2}$$

For $L>l$, this equation has one solution for $A>0$ and one solution for $A<0$, contrary to the claim in [1] that there is one solution for $A>0$ and two solutions for $A<0$. Thus the behaviour of the inextensible string is different from that of the elastic string, which will be discussed in what follows. Note that $A<0$ implies $\lambda<0$, because of $y(l)=a>0$, which means a negative tension.

The solution for $A<0$ is easily shown to be unstable on the basis of the energy criterion. Denote the integrand of (1) by $F(y, y')$, then a necessary condition of Legendre [7] for the potential energy to have a minimum is that $\partial^2F/\partial y'^2$is non-negative. From

$$F=(y+\lambda)\sqrt{1+y'^2}\,,\qquad \partial^2F/\partial y'^2=(y+\lambda)(1+y'^2)^{-3/2}$$

it is seen that this condition is violated for $y+\lambda<0$, that is, for $A<0$.

Remark. The catenary problem has an interesting two-dimensional analogue that goes back to Jellet [8] in 1850, but that has only recently been solved [9,10]:

$$\int_G\int(z+\lambda)\sqrt{1+z,_x^2+z,_y^2}\,\mathrm{d}x\mathrm{d}y=\text{stationary}.$$

3. The Föppl theory of the elastic string

Consider a linear elastic material, that is, the stress T is related to the strain ϵ by Hooke's law $T=E\epsilon$. The potential energy of a vertically loaded string of length $2l$ is then given by

$$V=\tfrac{1}{2}E\int_{-l}^{+l}\epsilon^2\mathrm{d}\xi-\int_{-l}^{+l}PW\mathrm{d}\xi, \tag{3}$$

where $P(\xi)$ is a vertical load (force per unit undeformed length divided by the cross-sectional area of the string) and W is the vertical displacement which is assumed to vanish at the ends,

$$W(l) = W(-l) = 0. \tag{4}$$

If U denotes the horizontal displacement, the basic assumptions of the Föppl approximate theory of strings and membranes are: $U \ll W$ and W small but finite. Thus the strain-displacement relation is $\epsilon = U' + \frac{1}{2}(W')^2$, the prime denoting differentiation with respect to ξ, and the energy functional (3) is

$$V[U, W] = \tfrac{1}{2}E\int_{-l}^{+l}\left[(U')^2 + U'(W')^2 + \tfrac{1}{4}(W')^4\right]d\xi - \int_{-l}^{+l}PW d\xi. \tag{3a}$$

Now $\delta V = 0$, for all admissible U, W, yields the Euler equations

$$\left(U' + \tfrac{1}{2}(W')^2\right)' = 0, \tag{5}$$

$$E\left(U'W' + \tfrac{1}{2}(W')^3\right)' + P = 0, \tag{6}$$

which can be re-written in the form

$$U' + \tfrac{1}{2}(W')^2 = \epsilon = \text{constant}, \tag{7}$$

$$E\left(U' + \tfrac{1}{2}(W')^2\right)W'' + P = 0. \tag{8}$$

From (7) we have ϵ and T constant, so that equation (8), $TW'' = -P(\xi)$ can be integrated, the two constants of integration being determined by (4). Substitution of W' into (7) yields U. For *uniform load* P = const the results are, with an arbitrary constant U_0

$$W(\xi) = \frac{P}{2T}(l^2 - \xi^2), \qquad U(\xi) = \frac{T}{E}\xi - \frac{P^2}{6T^2}\xi^3 + U_0. \tag{9}$$

At the ends $\pm l$, we may prescribe either the stress $T(\pm l) = T_0$ or the horizontal displacement $U(\pm l) = \pm N$. In the following we refer to these two boundary problems as *Problem S* or *Problem H*, respectively. From (8) and (9) we have immediately

Theorem 1. The solution of Problem S is unique for all $T_0 > 0$ and for all $T_0 < 0$, except for a uniform translation U_0. There is no solution for $T_0 = 0$ unless $P = 0$.

Imposing the boundary conditions $U(\pm l) = \pm N$, we find $U_0 = 0$ and $(l/E)T - P^2l^3/6T^2 = N$, which is a cubic equation for the unknown T. It can be rewritten in the form

$$F(A) := A(c_0 + A)^2 = \tfrac{1}{6}EP^2l^2, \qquad c_0 := \frac{NE}{l}, \qquad A := T - c_0. \tag{10}$$

Equations (9) and (10) agree with results of Dickey [1], who observed that Eq. (10) has a unique solution $A > 0$ if $c_0 \geqslant 0$, that is for $U(l) \geqslant 0$, and that Eq. (10) may have three solutions if $c_0 < 0$ and P is sufficiently small.

In order to describe the range of non-uniqueness more precisely and to display the dependence of the solution on a single parameter δ, it is appropriate to introduce the dimensionless variables (for Problem H)

$$\bar{w}=\frac{W}{l}\left(\frac{Pl}{E}\right)^{-1/3},\qquad \bar{u}=\frac{U}{l}\left(\frac{Pl}{E}\right)^{-2/3},\qquad x=\frac{\xi}{l},\qquad \delta=\frac{N}{l}\left(\frac{Pl}{E}\right)^{-2/3}.$$

Then the solution (9) and (10) can be written in the form

$$\bar{A}^{-1/2}\bar{w}=(3/2)^{1/2}(1-x^2),\qquad \bar{A}^{-1}\bar{u}=\bar{\delta}x-x^3,\qquad \bar{\delta}=1+(\delta/\bar{A}),\tag{9a}$$

where the positive constant $\bar{A}$ has to be determined from

$$\bar{F}(\bar{A}):=\bar{A}(\delta+\bar{A})^2=\tfrac{1}{6}.\tag{10a}$$

It is seen that the dimensionless solution $\bar{u}$, $\bar{w}$ depends on δ only, that is, essentially on the ratio $U(l)/P^{2/3}$. Now, if $\delta<0$, we find from (10a) one, two or three solutions for $\bar{A}$ depending on whether $|\delta|$ is less, equal or greater than $(9/8)^{1/3}$, respectively. We summarize the results in

Theorem 2. Consider Problem H of the Föppl approximation for the elastic string under a constant vertical load P. If $\delta\geqslant 0$, there is a unique solution for all P given by (9) with $U_0=0$, $T=c_0+A$, or by (9a), with A or $\bar{A}$ uniquely determined from (10) or (10a), respectively. The solution is *tensile*, that is, $T>0$. – If $\delta<0$, Problem H has a unique tensile solution for all P, with A or $\bar{A}$ now being defined as the largest root of the cubic equation (10) or (10a). In addition, there exist two (or one) *compressive* solutions $T<0$, if and only if $|\delta|$ is greater than (or equal) $(9/8)^{1/3}$. These solutions are again given by (9) or (9a), where A or $\bar{A}$ is one of the two smaller roots of (10) or (10a), respectively.

Next we show that all solutions with $T<0$ are unstable according to the energy criterion. Denoting the integrand of (3a) by $2F(U, W, U', W')$, the Legendre condition used above now requires that the quadratic form of the matrix

$$\begin{pmatrix} F_{,U'U'} & F_{,U'W'} \\ F_{,U'W'} & F_{,W'W'} \end{pmatrix}$$

is non-negative. A comma denotes partial differentiation. From

$$F_{,U'U'}F_{,W'W'}-F^2_{,U'W'}=U'+\tfrac{3}{2}(W')^2-(W')^2=\epsilon=T/E$$

it follows that Legendre's condition is violated for $T<0$, hence all compressive solutions described in Theorems 1 and 2 are unstable.

It is of interest to compare the potential energies in the range $-\delta>(9/8)^{1/3}$, where three solutions of Problem H exist. Substituting the solution into (3) and using (10) and (10a), we obtain for the potential energy

$$V(P,N)=\frac{1}{4c_1}(c_0+A)(c_0-3A)=lE\left(\frac{Pl}{E}\right)^{4/3}(\delta+\bar{A})(\delta-3\bar{A}).\tag{11}$$

For given P and $c_0 = NE/l$, A is determined from (10), hence $V(P, N)$ is not single-valued if $(-\delta) > (9/8)^{1/3}$, because (10) has three solutions in that case.

Theorem 3. For $\delta < 0$, $|\delta| > (9/8)^{1/3}$, denote the solutions of (10) by A_i, $i = 1, 2, 3$, with $A_1 < A_2 < A_3$, and the corresponding energies $V(P, N)$ by V_i. Then $V_1 > V_2 > V_3$; for the compressive solutions $V_1 > 0$, $V_2 > 0$ holds, for the tensile solution $V_3 < 0$.

Proof. The functions $F(A) = A(c_0 + A)^2$ and $V(A) = (c_0 + A)(c_0 - 3A)/4c_1$ are sketched in Fig. 1 for $c_0 < 0$. Since both F and V have a maximum at $A = -c_0/3$, and V is symmetric with respect to the vertical line $A = -c_0/3$ while F is not, it is a simple matter to verify that the segments $a = -(c_0/3) - A_1$ and $b = A_2 + (c_0/3)$ in Fig. 1 satisfy $a < b$, so that we have indeed $V_1 > V_2 > 0$ from the intersections $A = A_i$ with $V(A)$. $A_3 > -c_0$ implies $V_3 < 0$. There is no doubt that for $(-\delta) > (9/8)^{1/3}$ the solution corresponding to A_3 is stable.

The solution (9) of Problem H has the symmetry property

$$U(0) = 0, \qquad W'(0) = 0. \tag{12}$$

The same conclusion can be verified for a *variable load* $P(\xi) = P(-\xi)$. Integrating $TW'' = -P(\xi)$, with $P(\xi)$ piecewise continuous in $[-l, l]$, and imposing the boundary conditions $W(\pm l) = 0$, we first obtain

$$W(\xi) = -\frac{1}{T}\int_{-l}^{\xi} Q(s)\mathrm{d}s + \frac{1}{2lT}(\xi + l)\int_{-l}^{l} Q(s)\mathrm{d}s, \qquad Q(s) := \int_{-l}^{s} P(\xi)\mathrm{d}\xi. \tag{13}$$

Substitution of $W'(\xi)$ into $U' + (W')^2/2 = \epsilon$ and integration yields

$$U(\xi) = U_0 + \frac{T}{E}(\xi + l) - \frac{1}{2T^2}\left\{\int_{-l}^{\xi} Q^2(s)\mathrm{d}s + \left[(\xi + l)\bar{Q} - 2\int_{-l}^{\xi} Q(s)\mathrm{d}s\right]\bar{Q}\right\} \tag{14}$$

with $\bar{Q} = (1/2l)\int_{-l}^{+l} Q(s)\mathrm{d}s$.

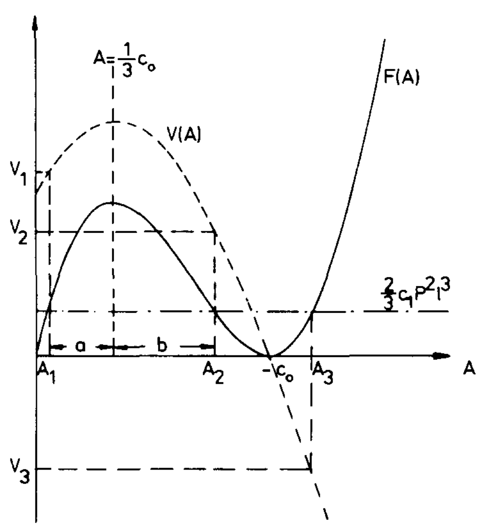


Fig. 1. Föppl approximation: the functions $F(A)$ and $V(A)$ for $c_0 < 0$ ($\delta < 0$), Problem H.

Equations (13) and (14) represent the solutions of Problems S and H for an *arbitrary vertical load* $P(\xi)$. In Problem S, we have $T = T_0(\pm l)$. In Problem H, $U(-l) = -N$ implies $U_0 = -N$, while $U(l) = N$ is satisfied, if T is determined again from a cubic equation:

$$N = \left(\frac{l}{E}\right)T - \frac{l^2}{T^2}\left(\overline{Q^2} - \frac{1}{2l}\overline{Q}^2\right), \qquad \overline{Q^2} = \frac{1}{4l^2}\int_{-l}^{l} Q^2(s)\mathrm{d}s. \tag{15}$$

From the Schwarz inequality it follows that $\overline{Q^2} - \overline{Q}^2/2l =: K^2$ is nonnegative, hence (15) can be written in the form $A(c_0 + A)^2 = lEK^2$, with c_0 and A as defined in (10). Consequently, we have

Theorem 4. For variable load $P(\xi)$, piece-wise continuous for $-l \leqslant \xi \leqslant l$, Theorems 1 and 2 remain valid: there is a unique solution of Problem S for $T_0 \neq 0$; there is a unique tensile solution of Problem H for $c_0 \geqslant 0$ and $c_0 < 0$. In addition, two (or one) compressive solutions exist if and only if $c_0 < 0$ and $|\delta_0| \geqslant (9/8)^{1/3}$, where $\delta_0 := (N/l)(6lK^2/E^2)^{-1/3}$.

The compressive solutions are again unstable by the Legendre condition. Calculating W' from (13) and observing that $P(\xi) = P(-\xi)$ implies, upon interchanging the order of integration,

$$\int_{-l}^{l} Q(s)\mathrm{d}s = 2l\int_0^l P(s)\mathrm{d}s,$$

one finds $TW' = -Q(\xi) + Q(0)$, and with a similar calculation for $U(\xi)$, one has again the symmetry property (12) of Problem H for variable load in the case $P(\xi) = P(-\xi)$.

Remark. Integration of (7) from $-l$ to $+l$, substitution into (8), and using $U(\pm l) = \pm N$, yields a nonlinear differential equation for $W(\xi)$,

$$\left(c_0 + c_1\int_{-l}^{+l}(W')^2\mathrm{d}\xi\right)W'' = -P, \qquad c_0 = NE/l, \qquad c_1 = E/4l, \tag{16}$$

from which the solution (9), Problem H, was derived in [1].

4. Exact theory of elastic strings under a vertical load

In the exact nonlinear theory of strings, the displacements U, W are not restricted in magnitude and the stress T is related to the strain e by an arbitrary constitutive equation $T = f(e)$. Replacing the Föppl approximation by the exact relation $\epsilon = U' + (U'^2 + W'^2)/2$ in (3), $\delta V = 0$ yields the Euler equations

$$2U' + W'^2 + 3U'^2 + U'^3 + U'W'^2 = \text{constant},$$

$$E(2U'W' + W'^3 + U'^2W')' + 2P = 0,$$

which are considerably more complex than Eqs. (7) and (8), and are not a suitable set of equations, even for a linear-elastic material.

Let $S(e)$ be an elastic potential such that $T = \mathrm{d}S/\mathrm{d}e = f(e)$ ($S = Ee^2/2$ for a linear material). Following [2,3], we here use the strain measure

$$e = \sqrt{(1 + U'(\xi))^2 + W'(\xi)^2} - 1 \tag{17}$$

where ξ is again the arc length of the undeformed string. The potential energy is now

$$\bar{V} = \int_{-l}^{l} S(e)\mathrm{d}\xi - \int_{-l}^{l} PW\mathrm{d}\xi. \tag{18}$$

Introduce the new variables

$$x = \xi/l, \qquad u = U/l, \qquad w = W/l, \qquad p = Pl, \tag{19}$$

and let θ be the angle between the tangent of the deformed string and the horizontal, then the compatibility relations are

$$\cos\theta = \frac{1 + u'(x)}{1 + e}, \qquad \sin\theta = \frac{w'(x)}{1 + e}. \tag{20}$$

From Eqs. (17)–(19) we obtain

$$e = \sqrt{(1 + u')^2 + (w')^2} - 1, \qquad V = \bar{V}/l = \int_{-1}^{+1} (S(e) - pw)\mathrm{d}x. \tag{21}$$

Henceforth, the prime denotes differentiation with respect to x. The equilibrium equations follow from $\delta V[u, w] = 0$. Using the relation

$$\delta e = \frac{\delta e}{\delta u'}\delta u' + \frac{\delta e}{\delta w'}\delta w' = \cos\theta\delta u' + \sin\theta\delta w', \tag{22}$$

we find, upon integration by parts,

$$(T\cos\theta)' = 0, \qquad (T\sin\theta)' = -p, \tag{23}$$

which can also be derived directly, as in [3].

The solution for a *uniform load* p can be shown to be symmetric, that is, $u(0) = w'(0) = 0$. This solution, satisfying Eqs. (20) and (23) $p = \text{const}$, and $w(\pm 1) = 0$, was given in [1]:

$$T^2 = B^2 + p^2x^2, \qquad \cos\theta = \frac{B}{\sqrt{B^2 + p^2x^2}}, \qquad u = -x + B\int_0^x \frac{1 + e}{T(s)}\mathrm{d}s,$$

$$w = \int_x^1 \frac{1 + e}{T(s)} sp\mathrm{d}s. \tag{24}$$

A solution of Problem S is obtained, if the constant of integration B is determined from $T(\pm 1) = T_0$, while Eqs. (24) represents a solution of Problem H, $u(\pm 1) = \pm\nu$, provided B can be determined from the nonlinear equation

$$\nu + 1 = B\int_0^1 \frac{1 + e}{T}\mathrm{d}s = :I(B, p). \tag{25}$$

For the first problem, we have immediately from (24) and $B = +(T^2(1) - p^2)^{1/2}$ (choice of the negative sign will not affect T and w, while u_+ and u_- yield the same deformed shape of the string, because of $u_+ + u_- = -2x$, hence it suffices to take $B \geqslant 0$):

Theorem 5. Let $|T_0| \geqslant p$, with a *constant vertical load* p, then *problem* S has a unique solution, which is tensile (compressive) for $T_0 > 0 (T_0 < 0)$. If $|T_0| < p$, Problem S has no solution. The same holds true for a *variable load* $p(x)$, if p is replaced by $|\int_0^1 p(s)\mathrm{d}s|$. The solution is given by (24), with px replaced by $\int_0^x p(s)\mathrm{d}s$ in the case of a variable load.

Remark. The second part of the theorem actually refers to a symmetric variable load $p(-x) = p(x)$, implying $u(0) = w'(0) = 0$. But the results can be generalized to arbitrary loads $p(x)$ similarly as in the Föppl approximation, Eqs. (13) and (14).

The existence and multiplicity of solutions of Problem H was investigated by Dickey [1] for constant p, if $T = f(e)$ satisfies the following conditions: (i) $f(e)$ is defined for $-1 \leqslant e_- < e < e_+ < \infty$, (ii) $f(0) = 0$, (iii) $0 < f'(e) < \infty$. These conditions imply the existence of a unique inverse function $e = g(T)$ defined for $T_- < T < T_+$ with the properties $g(0) = 0$, $g'(T) > 0$, $g(T) > -1$ and $\lim_{e \to e_\pm} f(e) = T_\pm$. The results of [1] are summarized in

Theorem 6. Suppose $f(e)$ satisfies (i)–(iii), with $e_- = -1$, $e_+ = T_+ = \infty$, $T_- = -\infty$, and p is constant. Then there exists a unique tensile solution of *Problem H* for all ν, $-1 < \nu < \infty$. There are no compressive solutions for $\nu \geqslant 0$. If p is sufficiently small and $-1 < \nu < 0$, there exist *at least* two compressive solutions. If p is sufficiently large and $-1 < \nu < 0$, there are no compressive solutions.

Differentiating $I(B, p)$ defined in (25) twice with respect to B, we obtain

$$\frac{\partial^2 I(B,p)}{\partial B^2} = \int_0^1 \frac{B}{T^4}\left[3p^2 s^2\left(g'(T) - \frac{1+g}{T}\right) + TB^2 g''(T)\right]\mathrm{d}s.$$

If $T < 0$, one has $B < 0$, and from $f''(e) = -(f'(e))^3 g''(T) \geqslant 0$ it follows that $g''(T) \leqslant 0$, and therefore $\partial^2 I/\partial B^2 < 0$. Thus the conclusions in [1] can be strengthened to yield the

Corollary. In addition to the assumptions on $f(e)$ in Theorem 6, let $f(e)$ be convex $f''(e) \geqslant 0$, e.g. $f(e) = Ee$, then there exist exactly two compressive solutions, if p is sufficiently small and $-1 < \nu < 0$.

Note the remarkable similarity of the results of Theorems 2 and 6. By means of the Legendre condition, the instability of solutions with $T < 0$ can also be established:

Theorem 7. Let $f'(e) > 0$, then all compressive solutions described by (24) for Problems S and H with constant or variable load $p(x)$ are unstable.

Proof. As in Section 3, we calculate the discriminant

$$D := F_{,u'u'} F_{,w'w'} - F^2_{,u'w'} \qquad \text{for} \qquad F = S(e) - p(x)w(x)$$

in accordance with (20)–(22). We first obtain

$$F_{,u'u'}=\frac{1}{1+e}\left[\left(\frac{1+u'}{1+e}\right)^2(f'(e)(1+e)-f(e))+f(e)\right]$$

and two similar expressions for $F_{,w'w'}$ and $F_{,u'w'}$. Substituting these into D, we obtain after some algebra,

$$D=\frac{1}{1+e}f(e)f'(e), \qquad \text{that is,} \qquad D<0 \qquad \text{iff} \qquad ff'=Tf'<0,$$

completing the proof. The significance of condition (iii) on $f(e)$ as a material stability condition is recognized.

In the special case $T=f(e)=Ee$, p constant, the integrations in (24) can be carried out, resulting in the closed form exact solution, for positive T,

$$\begin{aligned} u(x)&=x\left(\frac{B}{E}-1\right)+\frac{B}{p}ln\left[\frac{1}{B}\left(px+\sqrt{B^2+p^2x^2}\right)\right],\\ w(x)&=\frac{1}{p}\left(\sqrt{B^2+p^2}-\sqrt{B^2+p^2x^2}\right)+\frac{p}{2E}(1-x^2), \end{aligned} \tag{26}$$

with $B=+\sqrt{T_0^2-p^2}$ for Problem S, and B determined from (25) for Problem H.

It is now of interest to compare the solutions of the exact theory with the Föppl approximation for a linear material $T=Ee$. In particular, we ask in which range of p does the Föppl solution represent a satisfactory approximation to the exact solution (26). For $\nu=O(1)$ we have $B=O(T)$ from (25). Expanding (26) with respect to p/B, we find, setting $b=B/E$

$$u(x)=bx-x^3\frac{p^2}{6B^2}+O\left(\left(\frac{p}{B}\right)^4\right), \tag{27}$$

$$w(x)=(1-x^2)(1+b)\frac{p}{2B}-(1-x^4)\frac{p^3}{8B^3}+O\left(\left(\frac{p}{B}\right)^5\right), \tag{28}$$

$$T(x)=B\left[1+x^2\frac{p^2}{2B^2}+O\left(\left(\frac{p}{B}\right)^4\right)\right]. \tag{29}$$

In the elastic range $T/E\ll 1$, whence $b\ll 1$. According to (29), the constant B approximates T with an error $O(p^2/B^2)$, while (27) and (28) imply, setting $T_0=T(0)=B$,

$$u(x)=ex-\frac{x^3p^2}{6T_0^2}+O\left(\frac{p^4}{T_0^4}\right), \qquad w(x)=(1-x^2)\frac{p}{2T_0}+O\left(\frac{p^3}{T_0^3}\right). \tag{30}$$

Expanding (17), one has $e=u'+(w')^2/2=\epsilon$. We conclude that the leading terms of the exact solution (30) coincide with the Föppl solution (9), which can be written, upon the change of variables (19), as

$$u(x)=\epsilon x-\frac{1}{6}\left(\frac{p}{T}\right)^2x^3, \qquad w(x)=\frac{p}{2T}(1-x^2). \tag{9b}$$

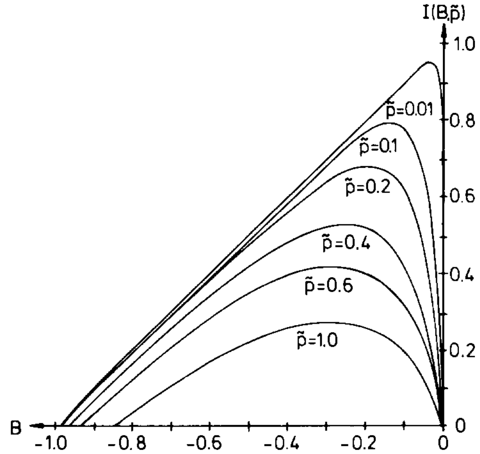


Fig. 2. Exact compressive solutions under vertical load: the function $I(B, \tilde{p})$ for $B < 0, f(e) = Ee$, Problem H.

Furthermore, quantitative estimates for the error, when the solution (26) is approximated by (9b), are provided by the expansions (27)–(29). In particular, if the constant T (Föppl) is replaced by $T = B(1 + x^2p^2/2B^2)$, then this modified Föppl approximation is accurate up to terms of $O(p^3/T_0^3)$ in $w(x)$, and up to terms of order $O(p^4/T_0^4)$ in $u(x)$ and $T(x)/B$. In view of the assumption $|u| \ll |w|$, the above comparison is valid only for sufficiently small values of $|\nu|$, especially in the range $T < 0$.

For the case of a linear material $f(e) = Ee$, the qualitative statement "p is sufficiently small (large)" in the second part of Theorem 6 can be made more precise by calculating the maxima of $I(B, p)$, $B < 0$ for a set of values p, see Fig. 2. For every ν, $-1 < \nu < 0$, there is a unique $p_c(\nu)$ such that two compressive solutions exist for all $p < p_c(\nu)$. The graph of $p_c(\nu)/E$ is shown in Fig. 3, together with the corresponding graph from the Föppl approximation $p_c(\nu)/E = \sqrt{8/9}\,|\nu|^{3/2}$, which follows from $|\delta| = (9/8)^{1/3}$ (Theorem 2).

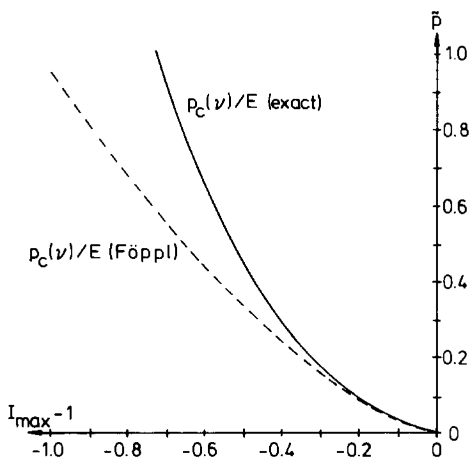


Fig. 3. Limiting load $\tilde{p}_c(\nu)$: region of compressive solutions $\tilde{p} < \tilde{p}_c(\nu)$, $-1 < \nu < 0$, exact theory (———) and Föppl approximation (- - - - -).

Conclusion. The (ν, p)-region of existence of compressive solutions, i.e. $p < p_c(\nu)$, in the exact theory with $f(e) = Ee$ is larger than in the Föppl theory, both regions are given in Fig. 3.

Remark. An example of a simple elastically nonlinear material, for which the integrations (24) can be carried out or reduced to standard elliptic integrals, is given in the Appendix.

It remains to discuss Problem H for a *variable load* $p(x)$ and without the restriction $e_- = 1$, $e_+ = \infty$, $T_\pm = \pm\infty$. In [2], Dickey has proved existence and multiplicity results, analogous to Theorem 6, under the assumption

$$\int_0^1 \frac{1}{|\tilde{p}(x)|} \mathrm{d}x < \infty, \qquad \tilde{p}(x) := \int_0^x p(x) \mathrm{d}x. \tag{31}$$

However, this assumption excludes practically all physically realistic loads such as $p(x)$ regular at $x = 0$, implying $\tilde{p}(x) = O(x)$ for $x \to 0$, so that $|1/\tilde{p}(x)|$ is not integrable. In order that (31) be satisfied, $p(x)$ has to be singular at $x = 0$. We therefore replace (31) by the following assumption

(i) $p(x)$ piece-wise continuous for $0 \leqslant x \leqslant 1$,
(ii) there exist constants $k > 0$, $\beta > 0$ and $\alpha \geqslant 1$ such that $|\tilde{p}(x)| \geqslant kx^\alpha$ for $0 \leqslant x \leqslant \beta$. (31a)

The solution of Problem H for nonuniform $p(x) = p(-x)$ is again given by Eqs. (24) and (25) except that $T^2 = B^2 + \tilde{p}(x)^2$ and that sp is replaced by $\tilde{p}(s)$ in the integral for $w(x)$. The uniqueness of tensile solutions follows from $\partial I(B, p)/\partial B > 0$ as before, independent of condition (31). The existence of solutions can be proved under the assumption (31a), by suitably modifying the proofs of some of the lemmas in [2]. We summarize the results (see [11]):

Theorem 8. Let $p(x) = p(-x)$ satisfy (31a). If $T_+ = \infty$, then a unique *tensile* solution of *Problem H* exists for all ν satisfying $0 < \nu + 1 < 1 + e_+$. If $T_+ < \infty$ and $\max|\tilde{p}(x)| =: \tilde{p}_M < T_+$, then a unique tensile solution exists for all ν satisfying $0 < \nu + 1 < I^+$, where

$$I^+ = \lim I(B, \tilde{p}) \qquad \text{as} \qquad B \to B^+ := \left(T_+^2 - \tilde{p}_M^2\right)^{1/2}.$$

Moreover, the deformed string $(x + u(x), w(x))$ is always convex (downward).

Theorem 9. Let $p(x) = p(-x)$ satisfy (31a). *Problem H* has no compressive solutions if $\nu \geqslant 0$. If $\tilde{p}_M$ is sufficiently small, then at least one compressive solution exists for all ν with $-1 < \nu < 0$; in the case $T_+ = e_+ = \infty$, $e_- = -1$, at least two compressive solutions exist, and if, in addition, $f''(e) \geqslant 0$, exactly two compressive solutions exist. On the other hand, if $\tilde{p}_M$ is sufficiently large, no compressive solutions exist with $1 < \nu < 0$. The deformed string is always concave (upward).

5. The elastic string under normal load

In this section solutions of the exact nonlinear equations of elastic strings under a constant normal load are obtained, and their multiplicity is discussed. The basic equations

(17) and (20) relating u, w, θ and e, as well as $T=f(e)$ remain the same. Let P_0 be the force per unit deformed length divided by the cross sectional area of the string, and directed perpendicular to the deformed string, then we have the equilibrium equations $(T\cos\theta)_s = P_0 \sin\theta$, $(T\sin\theta)_s = -P_0\cos\theta$, $s=$ arc length of the deformed string, or, in terms of the variables (19), with $\mathrm{d}s=(1+e)l\,\mathrm{d}x$, $p=P_0 l$,

$$(T\cos\theta)' = p(1+e)\sin\theta, \qquad (T\sin\theta)' = -p(1+e)\cos\theta. \tag{32a,b}$$

As the work done by the load is here $pw(1+u_x)l\,\mathrm{d}x$, the potential energy is now, as in (21) with $\mathrm{d}S/\mathrm{d}e=f(e)$,

$$V=\int_{-1}^{+1}(S(e)-pw(1+u'))\mathrm{d}x. \tag{33}$$

Using (22), $\delta V[u, w]=0$ yields the Euler equations (32).

The solution of (32) can be obtained in closed form. Carrying out the differentiations $(T\cos\theta)'$ and $(T\sin\theta)'$ in (32), multiplying (32a) by $\sin\theta$, (32b) by $\cos\theta$, and adding the resulting equations, we obtain

$$T\theta' = T\frac{\mathrm{d}\theta}{\mathrm{d}x} = -p(1+e). \tag{34}$$

Substituting (34) into (32), one finds $T'\cos\theta = T'\sin\theta = 0$, hence $T=T_0=f(e_0)$. Stress and strain are *constant* in elastic strings under normal pressure.

Equation (34) can now be integrated. Since the boundary conditions given below imply the symmetry conditions $u(0)=w'(0)=\theta'(0)=0$, we obtain $\theta(x)=-B_0x$, so that Eqs. (20) can be integrated with respect to x. Hence the solution of Eqs. (20) and (32) and $w(\pm 1)=0$ is given by

$$T=T_0=f(e_0), \qquad \theta=-B_0x, \qquad B_0=\frac{p(1+e_0)}{f(e_0)}, \tag{35a}$$

$$u=-x+C_0\sin B_0x, \qquad w=C_0(\cos B_0x-\cos B_0), \qquad C_0=\frac{1}{p}f(e_0). \tag{35b}$$

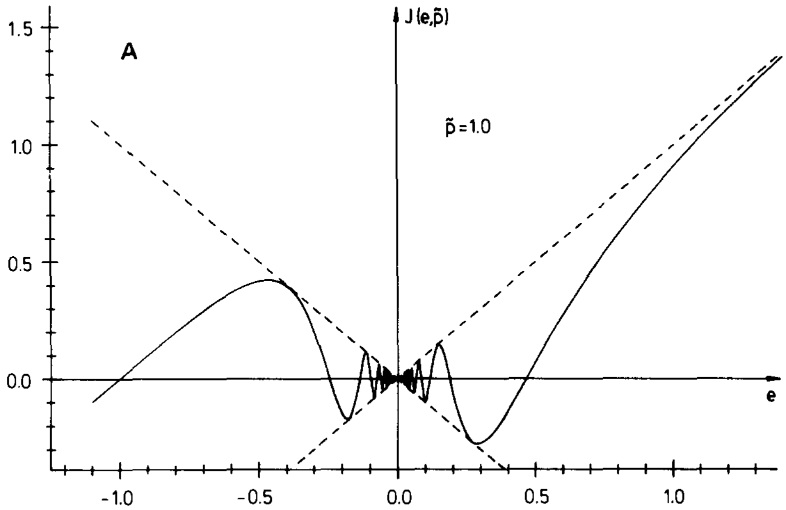
The constant T_0 (or e_0) is determined from the boundary conditions $T(\pm 1)=T_0$ in Problem S or $u(\pm 1)=\pm\nu$ in Problem H. In the first case we have

Theorem 10. *Problem S* for normal load has a unique solution for all values $T(1)=T_0\neq 0$, it is given by Eqs. (35) with $e_0=g(T_0)=f^{-1}(T_0)$. The solutions for $T_0<0$ are unstable.

Problem H has a solution if and only if e_0 satisfies $u(1)=\nu$, that is, e_0 is a solution of the equation

$$J(e_0, P):=\frac{F(e_0)}{P}\sin\left[\frac{P}{F(e_0)}(1+e_0)\right]=1+\nu \tag{36}$$

where $P:=p/E$ and $F(e):=f(e)/E$ are scaled by $E:=f'(0)>0$, implying $F(e)/e\to 1$ as $e\to 0$, and $P<1$ for any physically realistic material. It is seen that the solvability of (36)



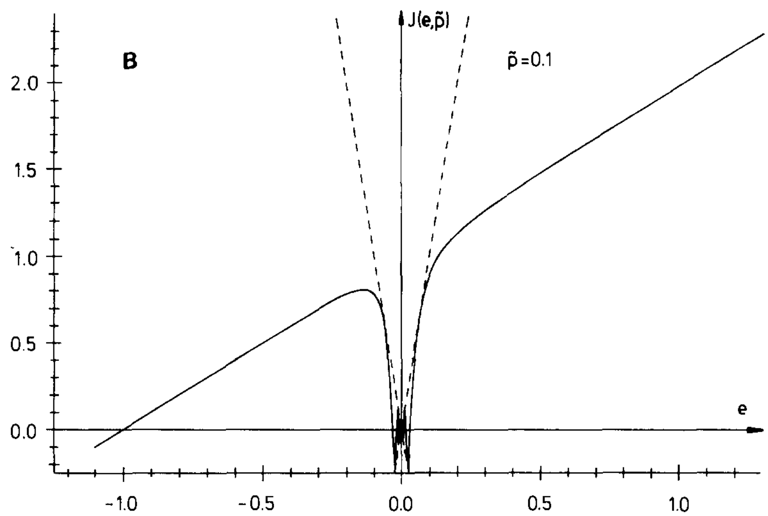


Fig. 4a. Normal load p, conservative: the function $J(e_0, \tilde{p}) = (e_0/\tilde{p})\sin(\tilde{p}e_0^{-1}(1+e_0))$ for $\tilde{p} = 1$.

Fig. 4b. Normal load p, conservative: the function $J(e_0, \tilde{p})$, as in Fig. 4a for $\tilde{p} = 0.1$.

depends on the properties of $F(e)$, where $f(e)$ is assumed to satisfy conditions (i)–(iii) of Section 4. There is a variety of possible cases, e.g. if T_+ is finite and if $|F(e)/P| < 1 + \nu$, Problem H has no solutions at all. On the other hand, Eq. (36) has infinitely many solutions $e_0^{(n)}$, with $\lim_{n\to\infty} e_0^{(n)} = +\infty$, for any given $P > 0$ and $\nu > -1$, if both $F(e)$ and $(1+e)/F(e)$ approach infinity with $e \to \infty$. In physical terms, situations like these usually correspond to prohibitively large p, ν or e, while the rate of increase of $f(e)$, as $e \to \infty$, is too low. However, for a wide class of functions $f(e)$, we now turn to show that Eq. (36) implies results similar to those illustrated in Fig. 4 for a linear material $f(e) = Ee$.

Assuming $e_+ = \infty$, define ϕ and c by

$$F(e) = e(1+\phi(e)), c(e) = \frac{1+e}{1+\phi(e)} > 0 \qquad \text{for} \quad e > 0$$

where $\phi(0) = 0$. For $e > 0$, we consider the following two cases.

(i) $c(e)$ remains bounded for all $e > 0$, that is, $0 < c(e) < c_1$. From (36), $J(e, P) = (e/P)(1 + \phi(e)) \sin(Pc(e)/e)$. As $c(e)/e \to 0$ with $e \to \infty$, we get, for sufficiently large e

$$J(e, P) = \frac{e}{P}(1+\phi(e))\left[\frac{P}{e}c(e) + O(e^{-3})\right] = 1 + e + r(e). \tag{37}$$

The leading term is $1 + e$, hence $\lim_{e\to\infty} J(e, P) = \infty$. Since $\lim_{e\to 0} J(e, P) = 0$, the equation $J(e, P) = 1 + \nu$ has at least one positive solution e_0 for any given ν, P with $\nu > -1$, $P > 0$. Moreover, in view of (37), the solution e_0 of (36) is unique if ν is sufficiently large.

(ii) $c(e)$ is unbounded, but $\lim_{e\to\infty} c(e)/e = c_2 \leqslant \pi$. In this case we have, for sufficiently large e

$$J(e, P) = \frac{e}{P}\cdot\frac{1+e}{c(e)} \sin(Pc_2 + \delta(e)), \qquad \delta(e) \to 0 \qquad \text{with} \qquad e \to \infty.$$

For $P < 1$, $\sin Pc_2 > 0$ for all e sufficiently large, hence we have

$$\lim_{e\to\infty} \frac{J(e, P)}{1+e} = \frac{\sin Pc_2}{Pc_2},$$

and we conclude as in case (ii), that Eq. (36) has at least one positive solution for any given ν, P with $\nu > -1$, $0 < P < 1$. The solution is also unique for ν sufficiently large.

Case (i) includes any functions ϕ with $\phi(e) = O(e^n)$ for $e \to \infty$, $n \geqslant 1$, case (ii) includes the linear material $\phi = 0$ or any function ϕ bounded for large e, e.g. $\phi(e) = (1 + e^2)^{-1}$. If $\phi(e) \geqslant 0$, one has $c_2 \leqslant 1$. Qualitatively similar results hold also for other cases, with suitable modifications, e.g. if $e_+ < \infty$ the range of ν must obviously be restricted for (36) to have solutions (as in Theorem 8).

The solutions of Eq. (36) for small $|e|$ behave like the solutions of the equation $x \sin(1/x) = 1 + \nu$, for any admissible $\phi(e)$, where $F(e)/e \to 1$ with $e \to 0$ has been used. This means that the graph of $J(e, P)$, P fixed, may intersect the line $1 + \nu$ several times if ν is sufficiently close to -1.

In the range $e_- < e < 0$, we consider the particular case $e_- = -1$ for which $J = 0$ and $\partial J/\partial e = 1$ at $e = -1$. As J returns to zero at $e = 0$, J must have at least one positive maximum in the interval $(-1, 0)$. Let $J_M(P) := \max J(e, P)$, e in $(-1, 0)$, then Eq. (36) has no solution $e_0 < 0$ for $1 + \nu > J_M(P)$, but at least two or one negative solutions, if $1 + \nu < J_M(P)$ or $1 + \nu = J_M(P)$, respectively. If P increases, $J_M(P)$ decreases, and vice-versa, that is, the larger P is, the smaller ν must be if negative solutions of (36) are to exist. Similar results hold for $e_- > -1$.

We summarize the main results in the following theorem, without repeating the restrictions imposed on $F(e)$ in cases (i) and (ii).

Theorem 11. Consider *Problem H* for normal load $P > 0$. There is a $P_0 > 0$ such that at least one *tensile* solution exists for $P < P_0$ and $-1 < \nu < \nu_0$, $\nu_0 = \infty$ in cases (i) and (ii). The solution is unique for $P < P_1 \leqslant P_0$ if $\nu > \nu_1 = \nu_1(P)$. If ν is sufficiently small, there exists more than one tensile solution, the number of solutions increases without limit as ν approaches -1.

For given P, there exist at least two *compressive* solutions for $-1 < \nu < \nu_c = \nu_c(P)$, the number of solutions increases without limit as ν approaches -1. The number ν_c decreases with increasing P. The solutions of Problem H, if they exist, are given by Eqs. (35), where e_0 is a solution of Eq. (36). The compressive solutions are unstable.

It is emphasized again, that Theorem 11 holds under certain assumptions on $f(e)$, somewhat more general than the ones considered above, but that part or all of Theorem 11 does not necessarily hold for some functions $f(e)$ with totally different properties.

It remains to show the instability of compressive solutions. The calculation of the discriminant D for $F = S(e) - pw(1 + u_x)$ is the same as in the proof of Theorem 7, the result is again $D = f(e)f'(e)/(1 + e) < 0$ for $T = f(e) < 0$, i.e. solutions with $e_0 < 0$ are unstable.

The content of Theorems 10 and 11 is to be compared with that of Theorems 5 and 6 for vertical load. In particular, we note the non-uniqueness of tensile solutions for *normal* load. In fact, we have an arbitrarily large number of both tensile and compressive solutions, if ν is sufficiently close to -1. This kind of multiplicity was found previously in [5] for compressive solutions of a circular elastic membrane under a *vertical* load for the two types of edge support corresponding to our problems S and H. There are also regions in the (ν, P)-plane where exactly one tensile and two compressive solutions exist, as in the case of a vertical load.

A consequence of (35b) is worth noting. In both problems S and H we have for the deformed position of the string $(X, Y) = (x + u, w)$

$$X^2 + (Y + Y_0)^2 = C_0^2 = (T_0/p)^2, \qquad Y_0 := C_0 \cos B_0.$$

Theorem 12. An elastic string under a uniform load always deforms into a circular arc, for any constitutive law $T = f(e)$. The radius of the circle is $R = |T_0|/p$, its center is $(0, Y_0)$, $C_0 > 0 (< 0)$ for tensile (compressive) solutions, but $\cos B_0 < 0$ for sufficiently small e_0.

The statements in Theorem 11 can be made more explicit for the case $T = f(e) = Ee$, assuming $P < \pi$. Eq. (36) then simplifies to

$$J(e_0, P) = \frac{e_0}{P} \sin \frac{P}{e_0}(1 + e_0) = 1 + \nu. \tag{38}$$

The following statements are easily verified. The function $J(e, P)$, with P fixed, oscillates between the pair of straight lines with slopes $\pm P^{-1}$ (see Fig. 4). The zeros of J in the range $e > -1$ are located at $e = e_n = P/(n\pi - P) > 0$ and at $e = e_{-n} = P/(P - \pi n) < 0$, $n = 1, 2, \ldots$ The extrema of J are located at $\tilde{e}_k = P/(z_k - P)$ where z_k are solutions of $\tan z = z - P$, which are approximately given by $\bar{z}_k = (2k + 1)\pi/2$, $k = \pm 1, \pm 2, \ldots$, the error for positive k is $\delta_k = \bar{z}_k - z_k > 0$ with $\delta_{k+1} < \delta_k$, $\delta_1 \leqslant 5 \cdot 10^{-2}$ for $P \leqslant 1$. The maxima $J^+_{M,k}$ (minima $J^-_{M,k}$) in the range $e > 0$ correspond to k positive and even (odd), they are given by

$$J^{\pm}_{M,k}(P) = \frac{\sin z_k}{(z_k - P)} = \cos z_k = \pm \frac{1}{\sqrt{1 + (z_k - P)^2}}. \tag{39}$$

Thus $J^+_{M,k} > 0$ and $J^-_{M,k} < 0$, the $J^+_{M,k}$ form a monotone decreasing null sequence. As $J(e, P)$ is monotone increasing for $e \geqslant P/(\pi - P) = e_1$ = maximal zero of J, it follows from (39) that (38) has a unique positive solution e_0, if $1 + \nu > J^+_{M,2}(P)$. It has exactly three positive solutions if $J^+_{M,4} < 1 + \nu < J^+_{M,2}$ and so forth. A rough estimate yields $J^+_{M,2} < 1/5$. The largest maximum in the range $e < 0$ is also obtained from (39), by setting $k = -1$ and taking the positive sign. Note that z_{-1} deviates from $\bar{z}_{-1} = -\pi/2$ for small P, while $z_k - \bar{z}_k > 0$ is small again for $k = -2, -3, \ldots$ We summarize these results in Theorem 13, replacing some qualitative statements of Theorem 11 by explicit estimates.

Theorem 13. Let $f(e) = Ee$ and $P < \pi$. Then Problem H has a unique tensile solution for $\nu > -4/5$, it has exactly $k = 2m + 1$, $m = 1, 2, \ldots$ tensile solutions if P and ν satisfy $J^+_{M,k+1} < 1 + \nu < J^+_{M,k-1}$, where $k \to \infty$ with $\nu \to -1$. There are no compressive solutions if $\nu > -1 + J^+_{M,-1}$, and there exist precisely $k = 2m$, $m = 1, 2, \ldots$ compressive solutions for P, ν satisfying $J^+_{M,-k+1} > 1 + \nu > J^+_{M,-k-1}$ where $k \to \infty$ with $\nu \to -1$.

The numbers $J^+_{M,\pm k}$ are given by (39), the solutions are given by Eqs. (35), where e_0 is a solution of (38).

Note that $\lim_{P\to\infty} J^+_{M,\pm k} = 0$ implies that no compressive solutions exist for a given ν, if P is sufficiently large.

The question may be raised, which one of several tensile solutions will be a preferred state of equilibrium, for given P and ν. Substituting the solution (35) into (33), the potential energy is

$$V(e_0) = 2S(e_0) - f(e_0)(1 + e_0)\left(1 - \frac{1}{2B_0} \sin 2B_0\right).$$

For solutions of Problem H, we get, upon inserting (36),

$$E^{-1}V(e_0) = S_1(e_0) + (1 + \nu)\left[F(e_0)^2 - P^2(1 + \nu)\right]^{1/2},$$

$$S_1(e) := 2E^{-1}S(e) - (1 + e)F(e).$$

Clearly $S_1'(e) = F(e) - (1 + e)F'(e)$, $S_1'(0) = -1$. For a wide class of functions $F(e)$, $S_1'(e) < 0$ at least in the range of e where multiple tensile solutions occur. Moreover, the term $-F(e)$ in S_1 will in general make $-F + (1 + \nu)[F^2 - P^2(1 + \nu)]^{1/2}$ negative, therefore $V(e_0) < V(e_0')$ for $e_0 > e_0' > 0$, e_0 and e_0' being two solutions of (36) for the same values of ν and P. In the special case $f(e) = Ee$, this can be verified explicitly from

$$E^{-1}V(e_0) = -e_0 + (1 + \nu)\left[e_0^2 - P^2(1 + \nu)\right]^{1/2},$$

as $1 + \nu < 1$ in the range of multiple tensile solutions. We conclude that, if Problem H has several tensile solutions for given ν, P, then the one with the *largest strain* e_0 has the *lowest potential* energy.

Next a different normal load problem is briefly discussed. If p is the force per unit *undeformed* length, as in Section 4, but directed perpendicular to the deformed string, one has the equilibrium equations,

$$(T \cos \theta)' = p \sin \theta, \qquad (T \sin \theta)' = -p \cos \theta. \tag{40}$$

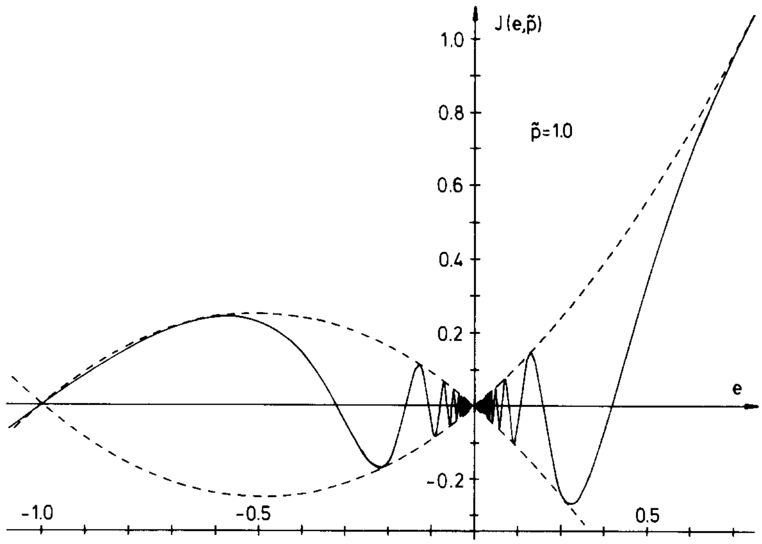


Fig. 5. Normal load p, nonconservative: the function $J(e_0, \tilde{p}) = (1 + e_0)(e_0/\tilde{p})\sin(\tilde{p}/e_0)$ for $\tilde{p} = 1$.

This is a type of follower-force problem which is apparently *non-conservative*, having no potential V similar to (33), with Eqs. (40) as Euler equations of $\delta V = 0$. Similar calculations lead again to $T\theta' = -p$ and $T = T_0 = f(e_0)$, a constant stress. Hence the solution is given by (35), except that $B_0 = p/f(e_0)$ and $C_0 = (1 + e_0)f(e_0)/p$, resulting in the following equation replacing (36);

$$J(e_0, P) = \frac{1 + e_0}{P} F(e_0) \sin \frac{P}{F(e_0)} = 1 + \nu. \tag{41}$$

Thus Theorems 10 and 11 are valid, except for different constants P_0, P_1, ν_0, ν_1 and ν_c and without the statement concerning the instability of compressive solutions. Theorem 12 remains true, for the new constants B_0, C_0 defined above. The arguments leading to Theorem 13 have to be slightly modified. $J(e, P)$ now oscillates between the pair of parabolas $y(e) = \pm(1 + e)e/P$ (see Fig. 5). The behavior of J for small e is the same as before, the values of $J^+_{M,k}$ differ but little from those given by (39). Hence Theorem 13 remains essentially valid.

6. Kinetic stability of the Föppl solution

The energy method is based on the observation that the transition from stability to instability is determined by the fact that the potential energy V ceases to be positive definite. This approach is restricted to conservative problems. It is not known, whether this static method generally yields the same results as the kinetic method for elastic continua, although it does so in many special cases. In the kinetic method, whose applicability is not restricted to conservative problems, the motion of a body in the vicinity of the equilibrium is investigated: A static solution is considered unstable, if the general free motion of the body in the vicinity of the static solution is not bounded, otherwise the solution is called stable (problems of hydrodynamic stability are generally investigated by this method).

The general approach in what follows is to start from the equations of motion of the string, and to consider solutions of the type $u_s(x)+\bar{u}(x, t)$, $w_s(x)+\bar{w}(x, t)$, where u_s, w_s stands for the static solutions under vertical or normal load given in the preceding sections. In accordance with the kinetic method, the equations of motion are linearized with respect to $\bar{u}$, $\bar{w}$ and motions of the type $\bar{u}=u(x)e^{i\omega t}$, $\bar{w}=w(x)e^{i\omega t}$ are investigated, from which the general free motion is obtained by superposition. The boundary conditions determine a set of frequencies ω_i. If $\omega_i^2>0$ for all i, $\bar{u}$, $\bar{w}$ are bounded for all t. If $\omega_i^2<0$ for some i, $\bar{u}$, $\bar{w}$ may increase exponentially with $t\to\infty$. Hence we employ a principle of linear stability.

Adding the inertia term ρW_{tt} to Eq. (16) of the Föppl approximation, where $W_t=\partial W/\partial t$, $\rho=$ linear material density of the string, assumed to be constant, we obtain the equation of motion

$$\left(c_0+c_1\int_{-l}^{+l}W_\xi(\xi, t)^2\mathrm{d}\xi\right)W_{\xi\xi}+P=\rho W_{tt}. \tag{42}$$

This is a special case of the nonlinear beam equation, for $P=0$,

$$\left(\beta+\gamma\int_0^l u_\xi(\xi, t)^2\mathrm{d}\xi\right)u_{xx}-\alpha u_{xxxx}=u_{tt}+\delta u_t, \qquad \alpha, \beta, \gamma, \delta>0,$$

studied by Ball [12] as an approximate model of the transverse motion of an elastic beam with fixed ends. Setting in (42) $W=W_s(\xi)+\overline{W}(\xi, t)$ yields, upon linearization in $\overline{W}$ and cancelling terms satisfying (16),

$$2c_1W_s''(\xi)\int_{-l}^{l}W_s'(\xi)\overline{W}_\xi(\xi, t)\mathrm{d}\xi+T\overline{W}_{\xi\xi}=\rho\overline{W}_{tt}. \tag{43}$$

Calculating W_s', W_s'' from (9), we obtain upon integrating by parts and using $\overline{W}(l, t)=\overline{W}(-l, t)=0$,

$$T\overline{W}_{\xi\xi}-\rho\overline{W}_{tt}=\bar{\beta}\int_{-l}^{l}\overline{W}(\xi, t)\mathrm{d}\xi, \qquad \text{and} \qquad \epsilon\hat{W}''+\Omega^2\hat{W}=\beta\int_{-l}^{l}\hat{W}(\xi)\mathrm{d}\xi \tag{44}$$

where $\overline{W}(\xi, t)=\hat{W}(\xi)\mathrm{e}^{i\omega t}$, $\bar{\beta}=2c_1(P/T)^2$, $\beta=P^2/(2lT^2)>0$, and $\Omega^2=\rho\omega^2/E$. The homogeneous equation (44) for $\hat{W}(\xi)$ together with $\hat{W}(\pm l)=0$ represents a (non-standard) eigenvalue problem. Setting

$$(\beta/\epsilon)\int_{-l}^{l}\hat{W}(\xi)\mathrm{d}\xi=\beta_1, \tag{45}$$

we distinguish two cases.

(i) $b^2:=\Omega^2/\epsilon>0$. The solution is $\hat{W}=C\sin b\xi+D\cos b\xi+\beta_1/b^2$. Imposing $\hat{W}(\pm l)=0$, we have either (ia) $C=0$, i.e. $bl\neq n\pi$, $D=-\beta_1/(b^2\cos bl)$, or (ib) $\sin bl=0$, i.e. $bl=n\pi$, $D=(-1)^{n+1}\beta_1/b^2$, $C\neq 0$.

The equation for the determination of the eigenvalues is now obtained by substituting

$\hat{W}$ into (45). In the subcase (ia) we find

$$\frac{1}{X}\tan X = 1 - m_1 X^2, \qquad X = l(\Omega^2/\epsilon)^{1/2} > 0, \qquad m_1 := \epsilon(T/Pl)^2 = (\delta + \bar{A})/6\bar{A}. \tag{46}$$

Equation (46) has an infinite number of solutions $X_m > 0$, with $X_m \sim (m - 1/2)\pi$, for $m \to \infty$. The eigenfunctions are

$$\hat{W}_m(\xi) = \cos X_m - \cos(X_m \xi/l) = \hat{W}_m(-\xi). \tag{47}$$

In the subcase (ib), the condition $b^2\epsilon = 2l\beta = (P/T)^2$ is obtained from (45). Since $b^2 = (n\pi/l)^2$, (ib) yields one eigenvalue if and only if P and T_0 (or ν) satisfy $(P/T)^2 = Tn^2\pi^2/(El^2)$, or equivalently $m_1 n^2 \pi^2 = 1$ for some integer n. In Problem S, $T = T_0$, while T is determined from Eq. (10) in Problem H. The eigenfunctions for this eigenvalue are

$$\hat{W}(\xi) = C_1[1 - (-1)^n \cos(n\pi\xi/l)] + C_2 \sin(n\pi\xi/l), \tag{48}$$

with C_1, C_2 arbitrary constants. Thus $\lambda = \Omega^2\epsilon^{-1} = (n\pi/l)^2$ is a double eigenvalue. If $C_2 \neq 0$, $\hat{W}'(0) \neq 0$, that is, $\hat{W}$ is in general unsymmetric.

(ii) $\Omega^2/\epsilon < 0$, $b := (-\Omega^2/\epsilon)^{1/2}$. The solution is $\hat{W} = C \sinh b\xi + D \cosh b\xi - \beta_1/b^2$. Imposing $\hat{W}(\pm l) = 0$, we have $C = 0$, $D = \beta_1/(b^2 \cosh bl)$. A similar calculation as in case (i) yields

$$\frac{1}{Y}\tanh Y = 1 + m_1 Y^2, \qquad Y = l(-\Omega^2/\epsilon)^{1/2}, \qquad m_1 = (\delta + \bar{A})/6\bar{A}. \tag{49}$$

Equation (49) has no solution if $m_1 \geqslant 0$, or if $m_1 \leqslant -1/3$, it has exactly one positive solution Y_0 if $-1/3 < m_1 < 0$, as will be shown below. In the latter case, there is a symmetric eigenfunction

$$\hat{W}_0(\xi) = \cosh Y_0 - \cosh(Y_0 \xi/l) = \hat{W}_0(-\xi). \tag{50}$$

For tensile solutions $T > 0$, we have $m_1 > 0$. Case (i) yields an infinite sequence $\Omega_m^2 = \epsilon(X_m/l)^2 > 0$, with $\lim \Omega_m^2 = \infty$, and one additional eigenvalue $\Omega^2 = \epsilon(n\pi/l)^2$ if $m_1 = (n\pi)^{-2}$, n integral. Case (ii) yields no solution. Thus all frequencies ω_m are real positive. For compressive solutions $T < 0$, we have $m_1 < 0$. Case (i) now yields a sequence $\Omega_m^2 = \epsilon(X_m/l)^2 < 0$, with $\lim \Omega_m^2 = -\infty$. Case (ii) yields one solution $\Omega_0^2 = -\epsilon(Y/l)^2 > 0$ if $-1/3 < m_1 < 0$, or no solution if $m_1 \leqslant -1/3$. As $\Omega_m^2 < 0$ implies $\omega_m = \pm i|\omega_m|$, admitting solutions $\hat{W}(\xi)\exp(|\omega_m|t) \to \infty$ with $t \to \infty$, we conclude kinematic instability. We summarize the results in

Theorem 14. On the basis of Eq. (42), the static solutions of the Föppl approximation for uniform load given in Section 3, are kinematically stable if T is positive, they are kinematically unstable if T is negative.

It remains to discuss the solvability of Eq. (49). Obviously, there is no solution $Y > 0$ if $m_1 \geqslant 0$. For $m_1 < 0$, Eq. (49) has no solution if and only if $1 + m_1 Y^2 < (\tanh Y)/Y$ for all

$Y > 0$. From the series expansion of tanh x we have

$$\frac{1}{x}\tanh x = \sum_{n=1}^{\infty}(-1)^{n-1}C_n x^{2n-2} = 1 - \frac{x^2}{3} + \left(C_3 - C_4x^2\right)x^4 + \left(C_5 - C_6x^2\right)x^8 + \dots \tag{51}$$

where $C_n = 2^{2n}(2^{2n}-1)B_{2n-1}/(2n)!$, B_{2n-1} being the Bernoulli numbers 1/6, 1/30, etc. From the classical formula

$$C_n = 2\left(\frac{2}{\pi}\right)^{2n}\left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \dots\right)$$

it follows that $C_{n+1}x^2/C_n < x^2(2/\pi)^2 \leqslant 1$, so that all terms in parentheses on the right hand side of (51) are nonnegative for $x \leqslant \pi/2$. Therefore,

$$\frac{1}{Y}\tanh Y \geqslant 1 - \tfrac{1}{3}Y^2 > 1 + m_1Y^2, \qquad \text{if} \qquad m_1 < -\tfrac{1}{3},$$

which actually holds for all $Y > 0$. On the other hand, if $m_1 > -1/3$, the derivative of $(1/Y)$ tanh Y near $Y = 0$ will be less than $2m_1Y$, so the parabola $1 + m_1Y^2$ must intersect $(1/Y)$ tanh Y once (the value $m_1 = -1/3$ in Problem H implies $\delta + 3\bar{A} = 0$, which also results from $\mathrm{d}[\bar{A}(\delta+\bar{A})^2]/\mathrm{d}\bar{A} = 0$, hence $m_1 = -1/3$ corresponds to the maximum of the function $F(A)$ in Fig. 1).

It is worth noting that both the energy criterion and the linear kinetic analysis yield the instability of solutions with $T < 0$. In the latter method, one positive eigenvalue Ω_0^2 is found if $-1/3 < \epsilon^2(Pl/E)^{-2} < 0$ in Problem S, and if $-1/3 < (\delta + \bar{A})/6\bar{A} < 0$ in Problem H. The occurrence of one double eigenvalue $\Omega^2 = (n\pi/l)^2$, in addition to the standard (Sturm-Liouville) eigenvalues is caused by the integral term in Eq. (44). It was only after completion of this work, that the authors became aware of a paper of Liouville [13] in 1837 on the solution of a similar integro-differential eigenvalue problem arising from thermodynamics.

In the preceding analysis, based on Eq. (42), a horizontal component of vibration $\bar{U}(\xi, t)$ has not been included. Although the Föppl approximation $U \ll W$ is retained, there is no a priori reason for ignoring $\bar{U}$ in a kinetic stability analysis, because both $\bar{U}$ and $\bar{W}$ are treated as small quantities, not excluding $\bar{U} = O(\bar{W})$. In order to examine the influence of $\bar{U}$, we proceed to carry out an exact (linear) stability analysis that accounts for both components $\bar{U}$ and $\bar{W}$.

Adding the inertia terms to the static equilibrium equations (5) and (6), we have the equations of motion for $U = U_s + \bar{U}$, $W = W_s + \bar{W}$

$$E\left(U_\xi + \tfrac{1}{2}W_\xi^2\right)_\xi = \rho U_{tt}, \qquad E\left(U_\xi W_\xi + \tfrac{1}{2}W_\xi^3\right)\xi + P = \rho W_{tt}. \tag{52}$$

Linearizing, cancelling static terms satisfying (5) and (6) and substituting U_s', U_s'', W_s', W_s'' from (9) for constant load P, we obtain, after some rearrangement of terms,

$$\bar{U}_{\xi\xi} - \bar{\gamma}\left(\xi\bar{W}_\xi\right)_\xi = (\rho/E)\bar{U}_{tt}, \qquad \left(\bar{q}(\xi)\bar{W}_\xi\right)_\xi - \bar{\gamma}\left(\xi\bar{U}_\xi\right)_\xi = (\rho/E)\bar{W}_{tt}, \tag{53}$$

where $\bar{q}(\xi) = \epsilon + \bar{\gamma}^2\xi^2$, $\bar{\gamma} = P/T_0$. Assuming $\bar{U} = \hat{U}(\xi)e^{i\omega t}$, $\bar{W} = \hat{W}(\xi)e^{i\omega t}$ and passing to the dimensionless variables (19), we obtain the following eigenvalue problem for $f(x) := \hat{U}(xl)/l$, $g(x) = \hat{W}(xl)/l$:

$$f'' + \lambda^2 f - \gamma(xg')' = 0, \qquad (q(x)g')' + \lambda^2 g - \gamma(xf')' = 0, \qquad -1 \leqslant x \leqslant 1, \tag{54}$$

where $\lambda^2 = \rho\omega^2 l^2/E$, $\gamma = \tilde{p}/\epsilon$, $\tilde{p} = Pl/E$, $q(x) = \epsilon + \gamma^2 x^2$. The boundary conditions are

$$f'(-1) + \gamma g'(-1) = f'(1) - \gamma g'(1) = g(-1) = g(1) = 0 \qquad \text{(Problem S)}, \tag{55a}$$
$$f(-1) = f(1) = g(-1) = g(1) = 0 \qquad \text{(Problem H)}. \tag{55b}$$

The former conditions are obtained from $\bar{T} = 0$ by substituting $W_s'(\xi)$ into

$$T = T_s + \bar{T} = E\left(U_s' + \bar{U}_\xi + \tfrac{1}{2}W_s'^2 + W_s'\bar{W}_\xi\right), \qquad \bar{T} = 0 = \bar{U}_\xi + W_s'\bar{W}_\xi,$$

the latter are obvious. A complete analysis of this eigenvalue problem has not been achieved as yet. A stability result will be given in Theorem 15 below.

Lemma A. Suppose $\epsilon \neq 0$, then the system (54) is *regular* for all x, $-1 \leqslant x \leqslant 1$.

Proof. Define $v = f'$, $z = g'$, and solve the linear equations (54) for $v' = f''$ and $z' = g''$, which is possible since the determinant is $q - \gamma^2 x^2 = \epsilon \neq 0$. The resulting first order system for $\mathbf{y} := (f, g, v, z)^T$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{y} - \mathbf{A}(x)\mathbf{y} = \lambda^2\mathbf{B}(x)\mathbf{y}, \qquad \mathbf{A} = \begin{pmatrix} 0 & I_2 \\ 0 & C_2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ E_2 & 0 \end{pmatrix}, \tag{56}$$

with

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2 = \frac{1}{\epsilon}\begin{pmatrix} x\gamma^2 & (\epsilon - x^2\gamma^2)\gamma \\ \gamma & -x\gamma^2 \end{pmatrix}, \qquad E_2 = -\frac{1}{\epsilon}\begin{pmatrix} q & x\gamma \\ x\gamma & 1 \end{pmatrix},$$

is equivalent to (54), the (4, 4)-matrices **A** and **B** are holomorph for all x. This can also be verified directly from (54) by expanding the solution f, g in series $r^\alpha \Sigma c_n r^n$ for $r = x$, or $r = x - x_0$, with $q(x_0) = 0$. The characteristic equation for α then yields $\alpha_1 = 0$, $\alpha_2 = 1$, and no terms $c \log|r|$. Hence, neither $x = 0$ nor $x = x_0$ are singular points of (54).

Lemma B. If $\epsilon \neq 0$, the eigenvalue problems (54) and (55) are selfadjoint. There exists a sequence of real eigenvalues λ_n^2, having no limit point except at infinity, and a complete set of orthogonal eigenfunctions. All λ_n^2 are positive if $\epsilon > 0$.

Proof. Introducing $\mathbf{z} = (f, g)^T$, the system (54) is equivalent to

$$\phi\mathbf{z} := -(D(x)\mathbf{z}')' = \lambda^2\mathbf{z}, \qquad D(x) := \begin{pmatrix} 1 & -\gamma x \\ -\gamma x & q(x) \end{pmatrix} = D(x)^T. \tag{57}$$

Let $\mathbf{z}$ and $\mathbf{w} := (\tilde{f}, \tilde{g})^T$ be twice differentiable, and let f, g and $\tilde{f}$, $\tilde{g}$ both satisfy the boundary conditions (55a) or (55b), then the usual integration by parts yields

$$(\mathbf{w}, \phi\mathbf{z}) := \int_{-1}^{+1} \mathbf{w}^T\phi\mathbf{z}\,dx = \int_{-1}^{+1}\left[f'\tilde{f}' + qg'\tilde{g}' - \gamma x\left(\tilde{f}'g' + f'\tilde{g}'\right)\right]dx = (\mathbf{z}, \phi\mathbf{w}). \qquad (58)$$

Since $\det D = q - \gamma^2x^2 = \epsilon \neq 0$, D is invertible for all x, and ϕ is selfadjoint. It follows from Lemma A that all solutions of $\mathbf{y}' - \mathbf{A}\mathbf{y} = 0$ are holomorph, the same is true for all solutions $\mathbf{z}$ of the equivalent equation $(D\mathbf{z}')' = 0$. Therefore, a continuous Green's matrix exists so that the eigenvalue problem (54) and (55) is equivalent to $\mathbf{z} = \lambda^2 K\mathbf{z}$, where K is a compact selfadjoint integral operator. The usual proof that such operators have a complete set of eigenfunctions with real eigenvalues carries over [7]. For two pairs of eigenfunctions (f_i, g_i), $i = 1, 2$, we have

$$\int_{-1}^{+1}\left[f_1(x)f_2(x) + g_1(x)g_2(x)\right]dx = 0, \qquad \text{if} \qquad \lambda_1^2 \neq \lambda_2^2. \qquad (59)$$

Let $\mathbf{z}$ be an eigenfunction, with eigenvalue λ^2, then we obtain from (57) and (58)

$$\lambda^2(\mathbf{z}, \mathbf{z}) = (\mathbf{z}, \phi\mathbf{z}) = \int_{-1}^{+1}(\mathbf{z}')^T D\mathbf{z}'\,dx = :Q(\mathbf{z}'). \qquad (60)$$

The eigenvalues of $D(x)$ are given by

$$\mu_{1,2} = \tfrac{1}{2}(1+q) \pm \tfrac{1}{2}\sqrt{(1+q)^2 - 4\epsilon} = \tfrac{1}{2}(1+q) \pm \tfrac{1}{2}\sqrt{(1-q)^2 + 4\gamma^2x^2}\,.$$

It follows that $\mu_1, \mu_2 > 0$ if and only if $\epsilon > 0$. In that case D is positive, the quadratic form $Q(\mathbf{z}')$ is positive definite implying $\lambda_n^2 > 0$ for all n. If $\epsilon < 0$, then $q + 1 = 1 - |\epsilon| + \gamma^2x^2 > 0$, since $|\epsilon| < 1$ in the Föppl approximation, and it follows that $\mu_1 > 0$, $\mu_2 < 0$ implying that $Q(\mathbf{z}')$ is indefinite, but the sign of $Q(\mathbf{z}')$, $\mathbf{z}$ an eigenfunction, cannot be determined. Thus we have

Theorem 15. On the basis of Eqs. (52), the static *tensile* solutions of the Föppl approximation for uniform load, given in Section 3, are kinematically *stable*, for *compressive* solutions the eigenvalue problem is indefinite.

This theorem remains valid for an arbitrary load $P(\xi)$. The calculations leading to Eqs. (53) can be performed by substituting into (52) $U_s', \ldots, W_s''$ from (13) and (14), resulting in

$$\overline{U}_{\xi\xi} - \left(\bar{r}(\xi)\overline{W}_\xi\right)_\xi = \rho E^{-1}\overline{U}_{tt}, \qquad \left(\bar{s}(\xi)\overline{W}_\xi\right)_\xi - \left(\bar{r}(\xi)\overline{U}_\xi\right)_\xi = \rho E^{-1}\overline{W}_{tt} \qquad (53a)$$

with $\bar{r}(\xi) = (Q(\xi) - \overline{Q})/T$, $\bar{s}(\xi) = \epsilon + \bar{r}(\xi)^2$. This leads to a selfadjoint eigenvalue problem as before, with γx and $q(x)$ in (54) replaced by functions $r(x)$ and $s(x)$, respectively, and $f'(\pm 1) \mp r(1)g'(\pm 1) = 0$ in (55a). Lemma A remains valid, as the determinant in the reduction to a first order system is again $s(x) - r^2(x) = \epsilon \neq 0$. Lemma B remains valid also, with γx and $q(x)$ in $D(x)$ replaced by $r(x)$ and $s(x)$, respectively. Thus ϕ is selfadjoint, it is positive if $\epsilon > 0$, whence we have

Theorem 16. On the basis of Eqs. (52) the static solution U, W given in (13) and (14) for *arbitrary* load $P(\xi)$ is kinematically stable if ϵ is positive.

In the case $\epsilon < 0$, it may be conjectured that both infinitely many positive and infinitely many negative eigenvalues occur. This is the case for scalar eigenvalue problems of the form

$$-(py')' + qy = \lambda ry, \qquad a \leqslant x \leqslant b, \qquad p(x) > 0, \qquad r(x) \text{ changing sign in } (a, b),$$

together with appropriate boundary conditions at $x = a, b$ to make the problem selfadjoint (see Kamke [14]). The equations (54) can be rewritten in the form

$$-\mathbf{z}'' + S\mathbf{z}' = \lambda^2 R\mathbf{z}, \qquad R = \epsilon^{-1}\begin{pmatrix} q & x\gamma \\ x\gamma & 1 \end{pmatrix} = R^T. \tag{61}$$

The eigenvalues r_1, r_2 of R are positive if $\epsilon > 0$, whereas $r_1 < 0$, $r_2 > 0$ if $\epsilon < 0$, $|\epsilon| < 1$. But we have not been able to prove a result analogous to Kamke's for the present problem.

In order to further examine the case $\epsilon < 0$ and to compare the results with those of the simplified analysis based on Eq. (42), it was decided to compute some eigenvalues numerically. We first observe from the form of Eqs. (54), that two modes of vibration occur: solving (54) for $0 \leqslant x \leqslant 1$ with $f(0) = g'(0) = 0$ and one of the end conditions at $x = 1$ yields the *symmetric* mode, with $f(-x) = -f(x)$, $g(-x) = g(x)$. A solution of (54) for $0 \leqslant x \leqslant 1$ with $f'(0) = g(0) = 0$ and (55a) or (55b) for $x = 1$ is defined as the *antisymmetric* mode, where $f(-x) = f(x)$, $g(-x) = -g(x)$.

The differential equations (54) and boundary conditions (55) are discretized by a method given in Varga [15], using central differences on a uniform mesh, setting $x_j = jh$ and $f_j = f(x_j)$, $g_j = g(x_j)$, $j = 0, 1, \ldots, m+1$. The set of difference equations is then

Table 1
Approximate eigenvalues (columns 1) based on (44), eigenvalues λ^2 based on (54) and (55b), symmetric mode (columns 2) and antisymmetric mode (columns 3), $\tilde{p}$: $= p/E$

	1	2	3		1	2	3
$\nu = 0$	0.182	0.182	0.219	$\nu = 0$	0.846	0.828	0.631
$\tilde{p} = 0.01$	0.585	0.539	0.936	$\tilde{p} = 0.1$	2.718	1.877	3.471
$\epsilon = 0.0255$	1.581	1.483	2.116	$\epsilon = 0.1185$	7.339	5.628	4.294
	3.091	2.905	2.836		14.350	11.139	8.210
	5.107	4.789			23.702	13.435	14.660
	1	2	3		1	2	3
$\nu = -0.2$	0.177	0.121	0.041	$\nu = -0.2$	1.035	0.878	
$\tilde{p} = 0.01$	0.521	0.370	0.234	$\tilde{p} = 0.01$	−0.191	−0.131	−0.045
$\epsilon = 0.0089$	0.997	0.738	0.546	$\epsilon = -0.0093$	−0.564	−0.403	−0.253
	1.306	1.004	0.982		−1.121	−0.807	−0.589
	1.824	1.260	1.538		−1.862	−1.341	−1.059
	1	2	3		1	2	3
$\nu = -0.2$	–	39.42	22.20	$\nu = -0.5$	2.189	1.72	6.34
$\tilde{p} = 0.01$	–	9.87	2.47	$\tilde{p} = 0.1$	−0.501	−0.62	−0.18
$\epsilon = -0.1996$	−0.490	−0.49	−1.97	$\epsilon = -0.062$	−1.280	−1.99	−1.23
	−4.432	−4.42	−7.86		−3.752	−4.02	−2.92
	−12.31	−12.26	−17.63		−7.421	−6.71	−5.28
	−24.13	−23.96	−31.23				−8.29

equivalent to a system of $N = 2m + 1$ linear equations $A\mathbf{y} = \lambda^2 h^2 E\mathbf{y}$, where $A = A^T$ and E are real (N, N)-matrices and $\mathbf{y}$ is an N-vector. E is diagonal with all elements 1 except the first one, which is 1/2. The matrix A is sepdiagonal, further details are found in a forthcoming report [16].

The eigenvalues λ^2 and eigenfunctions f_j, g_j (for uniform load) were calculated by the QR-method for mesh spacings $h = 1/20$ and $h = 1/40$. As a check, the resulting values for λ^2 and the initial values for (f, g, v, z) at $x = 0$ were taken to solve the system (56) by a high-accuracy Runge-Kutta method, the solution at $x = 1$ should then satisfy the boundary conditions (55) approximately. An additional check is provided by the orthogonality condition (59). With the ordering $|\lambda_i^2| \leqslant |\lambda_{i+1}^2|$ and $h = 1/40$, the first four eigenvalues, say, may safely be assumed to differ from the exact values by less than 1% (on the basis of the above checks and the $O(h^2)$ difference approximation employed).

In Table 1, the first few eigenvalues ϵX_m^2 from Eq. (46) (or (49)), based on the approximate model (42), are shown in columns 1, they refer to symmetric modes, because the case (ib) does not occur for the chosen values of $\tilde{p}$ and ν. The first few approximate

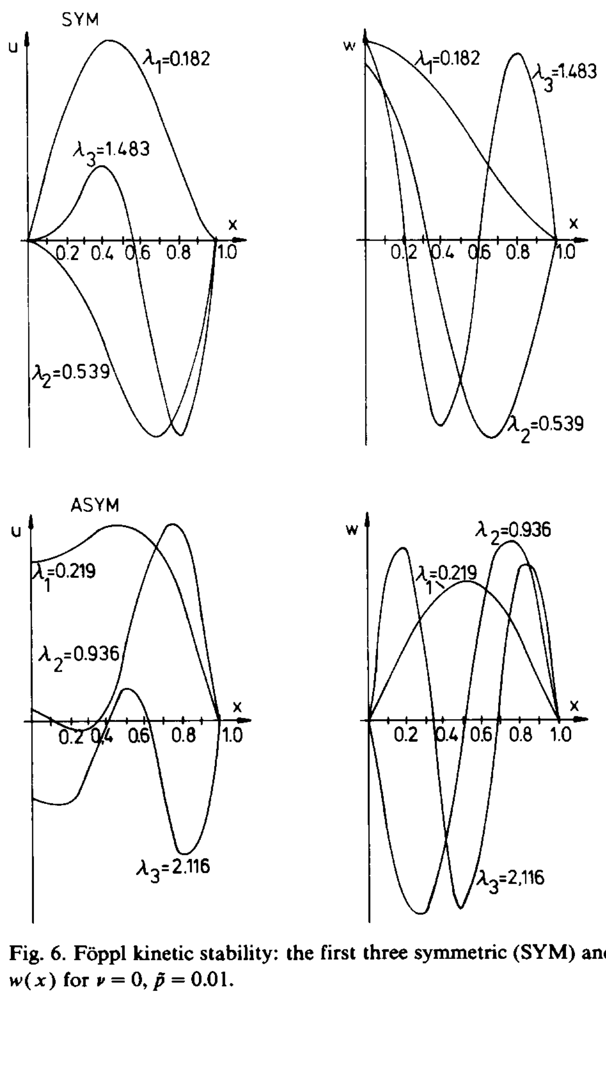


Fig. 6. Föppl kinetic stability: the first three symmetric (SYM) and antisymmetric (ASYM) eigenfunctions $u(x)$, $w(x)$ for $\nu = 0$, $\tilde{p} = 0.01$.

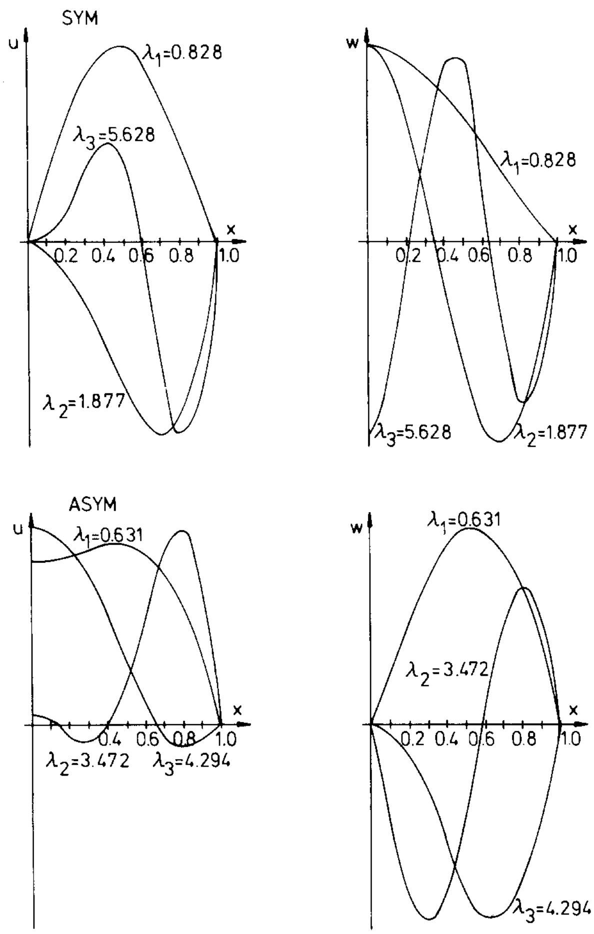


Fig. 7. Föppl kinetic stability: the first three symmetric (SYM) and antisymmetric (ASYM) eigenfunctions $u(x)$, $w(x)$ for $\nu = 0$, $\tilde{p} = 0.1$.

eigenvalues λ_m^2 for symmetric and antisymmetric modes, calculated by the preceding difference method, are displayed in columns 2 and 3, respectively. For $\nu = 0$ $\tilde{p} = 0.01$, there is fair agreement of columns 1 and 2, but for larger values of $\tilde{p}$ and $\nu \neq 0$, only qualitative common trends are observed, due to the approximation involved in model (42). The results also cast some doubt on the validity of the nonlinear beam model studied in [12]. In some cases $\epsilon < 0$, where Eq. (49) has one positive solution, one positive λ^2 is also found among the lowest few eigenvalues of (54) and (55). With increasing $-\epsilon$, no positive roots are found from (49), while the number of positive λ^2 found from the (discretized) equations (54) and (55) seems to increase, although the "density" of the negative eigenvalues is much higher.

In Figs. 6 and 7, the first three symmetric and antisymmetric eigenfunction pairs $(f(x), g(x))$, denoted again by (u, w) are plotted. In general, both the symmetric w and the antisymmetric u have no internal nodes for $\lambda^2 = \lambda_1^2$, the antisymmetric w and the symmetric u both have exactly one node at $x = 0$. The number of nodes of u and w is nondecreasing with increasing λ^2 (deviations from these trends were observed, see [16]).

7. Kinetic stability of the exact normal load solution

The equations of motion of the exact nonlinear theory of strings are obtained from the equations (23) and (32) of Sections 4 and 5 by adding the inertia terms:

$$(T\cos\theta)_x = \rho u_{tt}, \qquad (T\sin\theta)_x + p(x) = \rho w_{tt}, \tag{62}$$

$$(T\cos\theta)_x - p(1+e)\sin\theta = \rho u_{tt}, \qquad (T\sin\theta)_x + p(1+e)\cos\theta = \rho w_{tt}, \tag{63}$$

for vertical and constant normal load, respectively. Considering only the latter case in detail, we shall study the kinematic stability of the static normal load solutions given by (35) and (36). It turns out that a complete analytic solution of the stability equations can be found, while numerical methods must be employed in the case of vertical load.

Setting $T = T_s + \bar{T}(x, t)$, $\theta = \theta_s + \bar{\theta}(x, t)$, etc. in Eqs. (63), with $T_s = T_0 = f(e_0)$, $\theta_s = -B_0 x$ according to (35), and cancelling the static terms satisfying (32), the resulting equations are linearized with respect to the barred variables. Separating the time factor $e^{i\omega t}$ from $\bar{T}$, $\bar{\theta}$, $\bar{u}$, $\bar{w}$ and denoting the amplitudes again by T, θ, u, w, that is, $\bar{T} = T(x)e^{i\omega t}$ etc., the following equations are obtained

$$\begin{aligned} &cT' - T_0 s\theta' - B_1 sT + \rho\omega^2 u = 0, \qquad c := \cos\theta_s(x), \qquad s := \sin\theta_s(x),\\ &sT' + T_0 c\theta' + B_1 cT + \rho\omega^2 w = 0, \qquad B_1 := pf_0 - B_0, \qquad f_0 := 1/f'(e_0), \end{aligned} \tag{64}$$

where $f(e) = f(e_0 + \bar{e}) = f(e_0) + \bar{e}f'(e_0) = T_0 + \bar{T}$ has been used. Similarly, we get from (20) upon linearizing and cancelling static terms

$$T = \frac{1}{f_0}(cu' + sw'), \qquad \theta = \frac{1}{1+e_0}(cw' - su'). \tag{65}$$

Inserting these relations into (64) and simplifying yields, after some algebra,

$$\begin{aligned} &(P_1 u')' + (P_2 w')' - pf_0 w' + \rho\omega^2 f_0 u = 0,\\ &(P_2 u')' + (P_3 w')' + pf_0 u' + \rho\omega^2 f_0 w = 0, \end{aligned} \tag{66}$$

where $P_1(\mathrm{x}) = \alpha + \beta c^2$, $P_2(x) = \beta cs$, $P_3(x) = \alpha + \beta s^2$, $\alpha = pf_0/B_0$, $\beta = 1 - \alpha$. In Problem H, we have $u = w = 0$ at $x = \pm 1$ whereas $\bar{T} = w = 0$ in Problem S. Hence the boundary conditions are

$$u(\pm 1) = 0, \qquad w(\pm 1) = 0 \quad \text{(Problem H)}, \tag{67,H}$$

$$u'(\pm 1) \pm B_2 w'(\pm 1) = 0, \quad w(\pm 1) = 0 \quad \text{(Problem S)}, \tag{67,S}$$

where $B_2 = -\tan B_0$, which results from (65). Eqs. (66) and (67) constitute a linear eigenvalue problem for u, w and $\lambda := \rho\omega^2 f_0$. Setting $\mathbf{z} = (u, w)^T$, eqs. (66) can be written in matrix form

$$L\mathbf{z} := -\ (A(x)\mathbf{z}'\ )' - pf_0 B\mathbf{z}' = \lambda\mathbf{z}, \qquad A = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{68}$$

We note the close similarity between eqs. (54) and (66). In fact, Lemma A holds also for (66) if $e_0 \neq 0$, which is proved in exactly the same way, that is, by deriving a first order system of the form (56) with *regular* (4,4)-matrices $\mathbf{A}_0$, $\mathbf{B}_0$ equivalent to (66). However, due to the presence of the term Bz' in (68), we are not able to get a complete analogue of Lemma B. We do obtain self-adjointness for Problem H, but not for Problem S. In the former problem, we find $\lambda_n \geqslant -c$ for tensile solutions, but not $\lambda_n > 0$.

Theorem 17. For $e_0 \neq 0$, the eigenvalue problem (66) and (67,H) is selfadjoint, hence all eigenvalues λ_n are real. The set of eigenvalues forms a infinite sequence with no finite limit points, the associated eigenfunctions form a complete orthonormal system.

Proof. Let $\mathbf{z} = (u, w)^T$ and $\mathbf{v} = (\tilde{u}, \tilde{w})^T$, where both u, w and $\tilde{u}$, $\tilde{w}$ satisfy (67,H) and are twice differentiable, then we have, integrating by parts and using $A = A^T$

$$(\mathbf{v}, L\mathbf{z}) := \int_{-1}^{+1} \mathbf{v}^T L\mathbf{z} \mathrm{d}x = -\mathbf{v}^T A\mathbf{z}']_{-1}^{+1} + (\mathbf{v}', A\mathbf{z}') - pf_0 \int_{-1}^{+1} (\tilde{w}u' - \tilde{u}w')\mathrm{d}x$$

$$= (\mathbf{z}', A\mathbf{v}') + pf_0 \int_{-1}^{+1} (u\tilde{w}' - w\tilde{u}')\mathrm{d}x = (\mathbf{z}, L\mathbf{v}). \tag{69}$$

$A(x)$ is invertible for all x, as $\det A = \alpha = f_0 T_0/(1+e_0) \neq 0$. The self-adjointness of L now implies that all eigenvalues are real. In view of the regularity of the system (68), the existence of eigenvalues and eigenfunctions follows by the same argument as in the proof of Lemma B of Section 6, completing the proof of the theorem.

In Problem S, the above integration by parts, with eqs. (67,S), yields the boundary term,

$$\mathbf{v}^T A\mathbf{z}']_{-1}^{+1} = \tilde{u}(P_1 u' + P_2 w')]_{-1}^{+1}$$

$$= (P_2(1) - B_2 P_1(1))[\tilde{u}(1)w'(1) + \tilde{u}(-1)w'(-1)] \neq \mathbf{z}^T A\mathbf{v}']_{-1}^{+1}. \tag{70}$$

It is easily checked that $P_2(1) \neq B_2 P_1(1)$, whence problem (66) and (67,S) is not self-adjoint. We shall show below by a different method, that there exists an infinity of real eigenvalues, but we cannot exclude the occurrence of complex eigenvalues in Problem S.

Theorem 18. The eigenvalues λ_n of problem (66) and (67,H) for $e_0 > 0$ are bounded from below, that is, $\lim_{n\to\infty} \lambda_n = +\infty$. More precisely,

$$\lambda_n \geqslant m_1(e_0) - m_2(e_0), \qquad m_1 := \tfrac{1}{2}\pi^2 \min(1, \alpha) > 0, \qquad m_2 = 2\pi p f_0 > 0,$$

$$\alpha = \frac{f_0 T_0}{1 + e_0}. \tag{71}$$

If p, e_0 satisfy $\alpha < 1$ and $4p < \pi T_0/(1+e_0)$, then *all* eigenvalues are positive, and the static *tensile* solutions for normal load, given in Section 5, are kinematically *stable*.

Proof. Let $\mathbf{z}$ be a normalized eigenfunction with eigenvalue λ, then we have from (68),

$$(\mathbf{z}, L\mathbf{z}) = (\mathbf{z}', A\mathbf{z}') - pf_0(\mathbf{z}, B\mathbf{z}') = \lambda(\mathbf{z}, \mathbf{z}) = \lambda \int_{-1}^{+1} (u^2 + w^2)\mathrm{d}x = \lambda. \tag{72}$$

Calculating the eigenvalues a_i of $A(x)$, we find $a_1 = 1$, $a_2 = \alpha = f_0 T_0/(1 + e_0)$. Thus A is positive for $e_0 > 0$, and we have $(\mathbf{z}', A\mathbf{z}') \geqslant \min(1, \alpha)(\mathbf{z}', \mathbf{z}')$. The term involving B is estimated as follows, using the Schwarz inequality

$$|(\mathbf{z}, B\mathbf{z}')| \leqslant \int_{-1}^{+1} (|u'w| + |uw'|)\mathrm{d}x \leqslant \left(\int w^2 \mathrm{d}x \int u'^2 \mathrm{d}x\right)^{1/2} + \left(\int u^2 \mathrm{d}x \int w'^2 \mathrm{d}x\right)^{1/2}$$

$$\leqslant \left(\int (u^2 + w^2)\mathrm{d}x\right)^{1/2} (\|u'\| + \|w'\|) = \|u'\| + \|w'\| \leqslant \frac{4}{\pi}(\|u'\|^2 + \|w'\|^2) \tag{73}$$

where

$$\|u\| := \left(\int_{-1}^{+1} u^2 \mathrm{d}x\right)^{1/2}, \qquad \|u\|^2 + \|w\|^2 = 1,$$

and the following estimate has been used: Let $u(x)$, $v(x)$ be any functions vanishing at $x = \pm 1$ and satisfying $\|u\|^2 + \|v\|^2 = 1$, then, as will be proved below,

$$\|u'\|^2 + \|w'\|^2 \geqslant \frac{\pi}{4}(\|u'\| + \|v'\|). \tag{74}$$

Now we have $\|u'\|^2 + \|w'\|^2 = (\mathbf{z}', \mathbf{z}') \geqslant \pi^2(\mathbf{z}, \mathbf{z})/2 = \pi^2/2$, since $(\pi/2)^2$ is the smallest eigenvalue of $u'' + \lambda u = 0$, $u = 0$ at $x = \pm 1$. Collecting the preceding estimates and inserting them into (72), we obtain

$$\lambda \geqslant \left(\min(1, \alpha) - \frac{4}{\pi} p f_0\right)(\mathbf{z}', \mathbf{z}') \geqslant \tfrac{1}{2}\pi^2 \min(1, \alpha) - 2\pi p f_0,$$

the last term being positive if $\alpha < 1$ and $\pi\alpha > 4 p f_0$.

It remains to prove (74). From $(a + b)^2 \leqslant 2(a^2 + b^2)$ we find

$$2\frac{\|u'\|^2 + \|v'\|^2}{\|u'\| + \|v'\|} \geqslant \|u'\| + \|v'\| \geqslant \frac{\pi}{2}(\|u\| + \|v\|) \geqslant \frac{\pi}{2} \tag{75}$$

as $a^2 + b^2 = 1$ implies $a + b \geqslant 1$ for $a, b \geqslant 0$, and $u(1) = u(-1) = 0$ implies $\|u'\| \geqslant (\pi/2)\|u\|$ by the above argument. This completes the proof of the theorem.

Remark. In order to improve the estimate (71), more precise information on the value of $(\mathbf{z}, B\mathbf{z}')$ would be needed. The bound $m_1(e_0) - m_2(e_0)$ is positive for sufficiently small p, but will turn negative if e_0/p becomes small, as $m_1(0) = 0$. In the latter range, multiple tensile solutions may exist, according to Eq. (36). The actual occurrence of negative eigenvalues λ_n in this range would then cause some tensile solutions to be unstable.

The above appraisal for $(\mathbf{z}, B\mathbf{z}')$ holds independent of the sign of e_0. In the case $e_0 < 0$, $A(x)$ is indefinite ($a_1 = 1$, $a_2 = \alpha < 0$), but the sign of $(\mathbf{z}', A\mathbf{z}')$ in (72), depends on the range of $\mathbf{z}'$, which seems difficult to estimate a priori. Hence, instability of compressible solutions cannot be deduced from (72).

Recall that both Problems H and S are self-adjoint within the Föppl approximation $e_0 \ll 1$, $T_0 = E e_0$, $f_0 = E^{-1}$. The reason is that the term $p f_0 B\mathbf{z}'$ in (68) is negligible in this case. We then have $\sin\theta_0(x) \cong -B_0 x$, $\alpha = e_0/(1 + e_0) \cong e_0$, $\beta \cong 1$, $B_0 \cong p/T_0 = \gamma$, $p f_0 =$

p/E, and therefore, approximately

$$P_1 = 1, \qquad P_2 = -\gamma x, \qquad P_3 = e_0 + \gamma^2 x^2 = q(x),$$

in terms of the quantities defined following Eq. (54). For any elastic material $T \ll E$, which means $p/E \ll \gamma$. So the terms of $pf_0 u'$ and $pf_0 w'$ are indeed small compared with the other terms in eqs. (66). Furthermore, using $B_2 = -\tan B_0 = -\gamma$ in (70), we have $P_2(1) - B_2 P_1(1) = -\gamma + \gamma = 0$ in Problem S. Hence all non-selfadjoint terms disappear for small e_0.

Returning to the general case, we now proceed to give a different formulation of the stability problem, rather than attempting a direct analytic solution of eqs. (66). As the nonlinear eqs. (63) are rather complex, when expressed in terms of the variables u, w via (20) and (21), we eliminate u and w to get a more convenient set of the nonlinear dynamic equations in terms of the variables θ and T (or e). Differentiating (63) with respect to x and substituting from (20) for u_{xtt}, w_{xtt}, two equations are obtained, which are added (subtracted) after multiplying the first one by $\cos\theta$ ($\sin\theta$) and the second one by $\sin\theta$ ($\cos\theta$). The result is

$$\begin{aligned} &T_{xx} - T\theta_x^2 - pq\theta_x = \rho\left(q_{tt} - q\theta_t^2\right), \\ &T\theta_{xx} + 2T_x\theta_x + pq_x = \rho\left(q\theta_{tt} + 2q_t\theta_t\right), \qquad q := 1 + e = 1 + g(T), \end{aligned} \tag{76}$$

(see [17] for an equivalent transformation). For $T > 0$, (76) is a set of nonlinear hyperbolic equations in T and θ. Similar equations can obviously be derived from (62). The boundary conditions $w(1, t) = 0$, $u(1, t) = \nu$ now transform into nonlinear conditions for T and θ, namely

$$T(1, t)\theta_x(1, t) = p(1 + g(T)), \qquad \int_0^1 q\cos\theta \, dx = 1 + \nu$$

which result from (63) after setting $u_{tt} = w_{tt} = 0$ at $x = 1$ and eliminating $\sin\theta$, $\cos\theta$. For the linearized equations below, the boundary conditions are much simpler.

The stability equations are now obtained by setting $T = T_s + \bar{T}(x, t)$, $\theta = \theta_s + \bar{\theta}(x, t)$, with $T_s = T_0$, $\theta_s = -B_0 x$ as before, and linearizing (76) with respect to $\bar{T}$ and $\bar{\theta}$, which yields, upon cancellation of the static terms satisfying (32)

$$\begin{aligned} &\bar{T}_{xx} + B_0 B_1 \bar{T} + B_0 T_0 \bar{\theta}_x = \rho f_0 \bar{T}_{tt}, \\ &T_0 \bar{\theta}_{xx} + (B_1 - B_0)\bar{T}_x = \rho(1 + e_0)\bar{\theta}_{tt}, \end{aligned} \tag{77}$$

where B_1 and f_0 have been defined in (64). Separating $e^{i\omega t}$ from $\bar{T}$ and $\bar{\theta}$, and denoting the amplitudes again by $T(x)$ and $\theta(x)$, we have the following differential equations, which can also be derived from (64) and (65) by eliminating u and w,

$$\begin{aligned} &T''(x) + B_0 T_0 \theta'(x) + \left(B_0 B_1 + \rho\omega^2 f_0\right)T(x) = 0, \\ &T_0\theta''(x) + (B_1 - B_0)T'(x) + \rho(1 + e_0)\omega^2\theta(x) = 0. \end{aligned} \tag{78}$$

With one of the following two sets of boundary conditions

$$T'(\pm 1) = 0, \qquad \theta'(\pm 1) + (B_1/T_0)T(\pm 1) = 0 \quad \text{(Problem H)}, \tag{79,H}$$

$$T(\pm 1) = 0, \qquad \theta'(\pm 1) \pm (B_2/T_0)T'(\pm 1) = 0 \quad \text{(Problem S)}, \tag{79,S}$$

where $B_2 = -\tan B_0$, eqs. (78) define a linear eigenvalue problem for T, θ and ω^2. As the coefficients in (78) are constant, there is an elementary solution, together with a transcendental equation for ω^2. This is also true for the eigenvalue problem resulting from the normal load problem (40), but not for the vertical load stability problem, because T_s and θ_s given by (24) lead to linear equations for T and θ with variable coefficients, not solvable in closed form.

It remains to derive conditions (79). Inserting $T = T_s + \bar{T}, \ldots, w = w_s + \bar{w}$ into (63), linearizing and setting $x = 1$, $u = w = 0$ (Problem H), we obtain with $c_0 = \cos B_0$, $s_0 = -\sin B_0$,

$$c_0 T'(1) - s_0\big(B_1 T(1) + T_0\theta'(1)\big) = 0, \tag{80a}$$

$$s_0 T'(1) + c_0\big(B_1 T(1) + T_0\theta'(1)\big) = 0, \tag{80b}$$

and corresponding equations for $x = -1$. Since $c_0^2 + s_0^2 = 1$, $T'(\pm 1) = 0$ and $B_1 T(\pm 1) + T_0\theta'(\pm 1) = 0$ follow. As (80b) also holds in Problem S, we get the second condition of (79,S).

The static solution is symmetric because of $P = \text{const}$, thus the above solution T, θ splits into symmetric and antisymmetric modes in the way described in Section 6. Hence the eigenvalue problem is considered on $0 \leqslant x \leqslant 1$, with $u(0) = w'(0) = 0$ replacing the conditions (79) at $x = -1$ for symmetric modes, and similarly $u'(0) = w(0) = 0$ for antisymmetric modes. Starting from (63) and proceeding in the way just described, but setting $u = 0$ at $x = 0$, we obtain $T'(0) = 0$ instead of (80a), while (20) yields, upon linearization

$$\bar{e} \sin \theta_s(x) + (1 + e_0)\bar{\theta} \cos \theta_s(x) = \bar{w}_x,$$

so that $w'(0) = 0$ implies $\theta(0) = 0$. Similarly, we find $T(0) = \theta'(0) = 0$ in the asymmetric case. In summary, we have the following boundary conditions for Problems H and S:

$$(H)\ T'(1) = \theta'(1) + (B_1/T_0)T(1) = 0,\ T'(0) = \theta(0) = 0 \text{ for symmetric modes,} \tag{81}$$

$$(S)\ T(1) = \theta'(1) + (B_2/T_0)T'(1) = 0,\ T(0) = \theta'(0) = 0 \text{ for antisymmetric modes.} \tag{82}$$

The solution of (78) is obtained in the form $(T, \theta) = (E_0 A, C)e^{rx}$ with $E_0 := T_0/e_0$. The resulting linear equations for the coefficients A, C have nontrivial solutions iff their determinant vanishes. Setting $A = 1$, the solution of (78) reads

$$T/E_0 = \sum_{i=1}^{4} K_i e^{r_i x}, \qquad \theta = \sum_{i=1}^{4} K_i C_i e^{r_i x}, \qquad C_i = (-r_i e_0 B_0)^{-1}\big(r_i^2 + b_0 B_1 + \lambda f_0 E_0\big), \tag{83}$$

$$r^4 + b(e_0)r^2 + c(e_0) = 0, \qquad b(e_0) := \lambda(E_0 f_0 + k_0) + B_0^2, \tag{84}$$

$$c(e_0) := \lambda k_0(\lambda f_0 E_0 + B_0 B_1), \qquad \lambda = \rho\omega^2 E_0^{-1}, \qquad k_0 = 1 + e_0^{-1},$$

provided $e_0 \neq 0$ and the roots r_i, $i = 1, \ldots, 4$ of Eq. (84) are distinct. The coefficients K_i are to be determined from the boundary conditions (81) or (82).

The roots r_i depend on e_0 and λ. The discriminant of (84) can be written as

$$b(e_0)^2 - 4c(e_0) = \left[\lambda(E_0 f_0 - k_0) + B_0^2\right]^2 + 4\lambda k_0(B_0 - B_1) = :Q(\lambda). \tag{85}$$

Lemma C. The quadratic form $Q(\lambda)$, λ real, is positive definite if $e_0 < 0$. For $e_0 > 0$ it is indefinite, if $f(e_0)/f'(e_0) < 2(1 + e_0)$, otherwise it is positive definite. When $Q(\lambda)$ is indefinite, both zeros of $Q(\lambda) = 0$ are negative.

Proof. Rewriting $Q(\lambda)$ in the form $A\lambda^2 + 2B\lambda + C$, one finds

$$AC - B^2 = (E_0 f_0 - k_0)^2 B_0^4 - \left(2k_0 B_0(B_0 - B_1) + B_0^2(E_0 f_0 - k_0)\right)^2$$

$$= -4k_0^2 B_0^3(B_0 - B_1) = -4k_0^2 B_0^3\left[2(1 + e_0) - f(e_0)/f'(e_0)\right].$$

For $e_0 < 0$, T_0 and B_0 are negative, implying $AC - B^2 > 0$. For $e_0 > 0$, B_0 is positive, but the term in the brackets may be positive or negative as stated in the lemma. The real roots l_1, l_2 of $A\lambda^2 + 2B\lambda + C = 0$ satisfy $l_1 l_2 = C/A > 0$ and

$$-A(l_1 + l_2) = 4k_0 B_0(B_0 - B_1) + (E_0 f_0 - k_0)B_0^2 = 2k_0 B_0(2B_0 - B_1).$$

When $Q(\lambda)$ is indefinite, $e_0 > 0$ and $2B_0 - B_1 > 0$, consequently l_1 and l_2 are both negative.

Corollary. For $e_0 < 0$, the roots r_i of (84) are all distinct. The same holds for $e_0 > 0$ if $2(1 + e_0) < f(e_0)/f'(e_0)$.

Remarks. $Q(\lambda)$ is always positive for large $|\lambda|$. For $e_0 > 0$, there is a narrow range of λ, $l_2 < \lambda < l_1 < 0$, where $Q(\lambda) < 0$ if $f(e_0)/f'(e_0) < 2(1 + e_0)$ (which is always satisfied if $f(e_0) = Ee_0$, $e_0 > 0$), and there are double roots if $\lambda = l_1$ or l_2. These cases can occur if and only if p, e_0 are such that $\lambda \in [l_1, l_2]$ happens to be also a root of the transcendental equation that determines the eigenvalues λ_n (see eqs. (90)–(93) below).

The dependence of the roots r^2 of (84) on e_0 and λ can now be summarized as follows.

$e_0 > 0$ (1) $\lambda > 0$, implies $b^2 - 4c > 0$ and $b > 0$, with (a) $c < 0$, (b) $c > 0$.
(2) $\lambda < 0$, admits a range $b^2 - 4c < 0$, r^2 complex. If $b^2 - 4c > 0$, the subcases are (a) $c < 0$, (b) $c > 0$, $b > 0$, (c) $c > 0$, $b < 0$.
$e_0 < 0$ (3) $\lambda > 0$, implies $b^2 - 4c > 0$, with (a) $c < 0$, (b) $c > 0$, $b > 0$, (c) $c > 0$, $b < 0$.
(4) $\lambda < 0$, implies $b^2 - 4c > 0$, with subcases (a), (b) and (c) as in (3).

The signs of the roots in the three subcases are, denoting the two roots r^2 by m_1 and m_2:

$$(a)\ m_1 > 0, m_2 < 0 \qquad (b)\ m_1 < 0, m_2 < 0 \qquad (c)\ m_1 > 0, m_2 > 0.$$

In terms of the quantities r_i, C_i defined by (83), we have

$$r_1 = \sqrt{m_1} = -r_2, \qquad r_3 = \sqrt{m_2} = -r_4, \qquad C_1 = -C_2, \qquad C_3 = -C_4.$$

The list is to be completed by some obvious transition cases such as $c = 0$ in (1) with $m_1 = 0$, $m_2 < 0$, and by the case $b^2 = 4c$ for $e_0 > 0$, $\lambda < 0$, where $m_1 = m_2 = -b$. For materials $f(e) = Ee$ ($B_1 = -p/T_0$) the case (2a) cannot occur; in case (2) one has $b \neq 0$.

It remains to determine the constants K_i of (83). We shall consider one case in detail, it will be seen from the solution process that all other cases can be treated in exactly the same way. Consider Problem H for symmetric modes, and suppose that e_0, λ are such that subcase (a) applies, that is, $r_{1,2} = \pm\sqrt{m_1}$ and $r_{3,4} = \pm i\sqrt{-m_2}$. The boundary conditions $\theta(0) = T'(0) = 0$ of (81) are satisfied by setting $K_2 = K_1$, $K_4 = K_3$ (in all subcases (a), (b) and (c), and also in Problem S). Thus we have

$$\begin{pmatrix} T/E_0 \\ \theta \end{pmatrix} = \tfrac{1}{2}K_1\left[\begin{pmatrix} 1 \\ C_1 \end{pmatrix}e^{sx} + \begin{pmatrix} 1 \\ -C_1 \end{pmatrix}e^{-sx}\right] + \tfrac{1}{2}K_3\left[\begin{pmatrix} 1 \\ -C_3 \end{pmatrix}e^{i\beta x} + \begin{pmatrix} 1 \\ -C_3 \end{pmatrix}e^{-i\beta x}\right]$$

where $s := \sqrt{m_1}$, $\beta := \sqrt{-m_2}$. The remaining conditions of (81) yield

$$\begin{aligned} &sK_1 \sinh s - \beta K_3 \sin \beta = 0, \\ &(sC_1 + E_1)K_1 \cosh s + (i\beta C_3 + E_1)K_3 \cos \beta = 0, \qquad E_1 = E_0 B_1/T_0. \end{aligned} \tag{86}$$

The determinant of this system must vanish for solutions $(T, \theta) \neq (0, 0)$ to exist. Inserting the real constants C_1, iC_3 from (83), the following transcendental equation for the determination of the *eigenvalues* λ results:

$$s(\beta^2 - \lambda')\tanh s - \beta(s^2 + \lambda')\tan \beta = 0, \qquad \lambda' := \lambda f_0 E_0. \tag{87}$$

Recall that s and β are functions of λ. Solving (86), the *eigenfunctions* are, up to a constant

$$\begin{aligned} T(x) &= B_0 T_0(\beta \sin \beta \cosh sx + s \sinh s \cos \beta x), \\ \theta(x) &= -s^{-1}(s^2 + B_0 B_1 + \lambda')\beta \sin \beta \sinh sx + \beta^{-1}(\beta^2 - B_0 B_1 - \lambda')s \sinh s \sin \beta x. \end{aligned} \tag{88}$$

The transcendental equations for subcases (b) and (c) are

$$\beta_1 \tan \beta_1(\beta_2^2 - \lambda') = \beta_2 \tan \beta_2(\beta_1^2 - \lambda'), \quad s_1 \tanh s_1(s_2^2 + \lambda') = s_2 \tanh s_2(s_1^2 + \lambda'), \tag{89}$$

respectively, where $s_j = \sqrt{m_j} > 0$, $\beta_j = \sqrt{-m_j}$, $j = 1, 2$, with corresponding expressions for the eigenfunctions. Equations. (87) and (89) can be combined into the single equation (90) below. The transition case $c = 0$ in (1), $r_1 = r_2 = 0$ requires that all K_i are zero, unless $\beta = n\pi$, $n =$ integer. The double root case $m_1 = m_2$ could easily be treated by including terms Kxe^{rx} in the expressions for T and θ. Apart from these cases, which can occur only for exceptional values of p and ν, the solution of the eigenvalue problem (78) and (81) Problem H, is complete for real r^2, the eigenvalues λ_n must be found numerically from (90). The only eigenvalues not covered are those in the interval (l_1, l_2) for $e_0 > 0$, if there exist any (see Remarks following Lemma C).

The boundary conditions (82) at $x = 0$ for antisymmetric modes are satisfied by setting $K_2 = -K_1$, $K_4 = -K_3$, otherwise the calculations proceed in the same way, in both Problem H and Problem S. We list the results for the transcendental equations together with the case treated above, but omit listing the eigenfunctions analogous to (88).

Problem H, symmetric modes:

$$s(t^2 + \lambda') \tanh s - t(s^2 + \lambda') \tanh t = 0; \tag{90}$$

Problem H, antisymmetric modes:

$$s(t^2 + \lambda') \tanh t - t(s^2 + \lambda') \tanh s = 0; \tag{91}$$

Problem S, symmetric modes:

$$s^2 - t^2 = B_0 B_2 (s \tanh s - t \tanh t); \tag{92}$$

Problem S, antisymmetric modes:

$$s^2 - t^2 = B_0 B_2 (s \coth s - t \coth t), \tag{93}$$

where $s = s(\lambda)$, $t = t(\lambda)$, $s = \sqrt{m_1}$ if $m_1 > 0$, $s = i\sqrt{-m_1}$ if $m_1 < 0$, and $t = \sqrt{m_2}$ if $m_2 > 0$, $t = i\sqrt{-m_2}$ if $m_2 < 0$. Note that (90)–(93) are valid for $e_0 > 0$ and $e_0 < 0$, for arbitrary elastic material $T = f(e)$, and for all real $\lambda \neq 0$. For Problem S, only the *real* eigenvalues are given by Eqs. (92) and (93), except those in the interval $[l_1, l_2]$ for $e_0 > 0$. The analysis of the solutions of (84) is more involved for complex values of λ not excluded in Problem S.

Finally, we propose to analyze the preceding equations for large $|\lambda|$. The discussion will be carried out in detail for one particular case, the treatment of the remaining cases is analogous.

Consider Problem H for symmetric modes, $e_0 < 0$ and $\lambda < 0$. From (84) we have $c(e_0) < 0$ if $|\lambda|$ is sufficiently large, thus case (4a) with $m_1 > 0$, $m_2 < 0$ and (87) applies. We wish to show that (87) has infinitely many solutions $\lambda < 0$. For this purpose, rewrite (85) as

$$b^2 - 4c = b_1^2 \lambda^2 \left(1 + 2b_2 \lambda^{-1} + O(\lambda^{-2})\right), \qquad b_1 := f_0 E_0 - k_0 > 0,$$

for $|\lambda| \to \infty$, where b_2 is a constant independent of λ. With this, the following asymptotic expressions for the solutions r^2 of (84) are obtained

$$m_1 = F(\lambda) := -f_0 E_0 \lambda - \tfrac{1}{2}\left(B_0^2 + b_1 b_2\right) + O(\lambda^{-1}) = s^2 > 0, \tag{94}$$

$$m_2 = G(\lambda) := -k_0 \lambda - \tfrac{1}{2}\left(B_0^2 - b_1 b_2\right) + O(\lambda^{-1}) = -\beta^2 < 0. \tag{95}$$

Inserting these expressions into Eq. (87) we find, in terms of $g_1 = (f_0 E_0)^{1/2}$, $g_2 = (-k_0)^{1/2}$, and $X = |\lambda|^{1/2}$,

$$-g_1 X \tanh\left[g_1 X(1 + O(X^{-2}))\right] b_1 X^2(1 + O(X^{-2}))$$
$$+ g_2 X \tan\left[g_2 X(1 + O(X^{-2}))\right](B_0^2 + b_1 b_2)(\tfrac{1}{2} + O(X^{-2})) = 0 \tag{96}$$

for $X \to \infty$. Dropping all $O(X^{-2})$-terms in (96), we obtain

$$H(X) := X^2 \tanh g_1 X - g_3 \tan g_2 X = 0 \tag{97}$$

where $g_3 \neq 0$ is a constant. Observing that $X^2 \tanh(g_1 X)$ is regular for real x, its graph must intersect the graph of $\tan(g_2 X)$ infinitely many times. Furthermore, the function $H(X)$ takes on arbitrarily large positive and negative values between its zeros. Therefore, passing from (97) to (96) by including the terms of order $O(X^{-2})$ will not remove zeros if X is sufficiently large, and it follows that (96) has an infinity of roots $X_n > 0$.

A similar result holds for $e_0 < 0$, $\lambda > 0$. Again case (4a) applies for sufficiently large X, except that $m_1 = s^2 = G(\lambda)$ and $m_2 = -\beta^2 = F(\lambda)$, as defined by (94) and (95). An equation corresponding to (96) is obtained with g_1 and g_2, as well as $b_1 X^2$ and $(B_0^2 + b_1 b_2)/2$, interchanged, which yields, upon dropping $O(X^{-2})$-terms,

$$\tanh g_2 X - g_4 X^2 \tan g_1 X = 0. \tag{98}$$

The same reasoning as before leads to the conclusion that (98) and therefore (87) has infinitely many solutions $\lambda_n > 0$. Thus we have established

Theorem 19. The eigenvalue problem (78) and (79,H), $e_0 < 0$ has an infinity of positive and an infinity of negative eigenvalues. Hence the static *compressive* solutions of Problem H for normal load are kinematically *unstable*.

This result can be regarded as a counterpart of Theorem 18, observing that the two problems (66) and (67,H), and (78) and (79,H) are equivalent. The asymptotic method may also be used to verify the earlier results for $e_0 > 0$. If $\lambda < 0$, case (2c) with $c(e_0) > 0$, $b(e_0) < 0$ applies, which means that the λ_n are determined by Eq. (90) with both s and t real. Following the above procedure leads, up to terms of order $O(X^{-2})$, to the equation $\tanh g_2 X = CX^2 \tanh g_1 X$, $C \neq 0$, which has no real solutions for X sufficiently large. On the other hand, if $\lambda > 0$, case (1b) obviously applies, and eq. (90) leads to

$$\tan(g_2 X) - CX^2 \tan(g_1 X) = 0, \qquad g_1 \neq g_2, \qquad C \neq 0, \tag{99}$$

for large X. Equation (99) has infinitely many solutions, as will be shown below (Lemma D). Hence, the earlier results for $e_0 > 0$, Problem H are confirmed. In addition, the existence of *real* eigenvalues of the nonselfadjoint Problem S, which could not be deduced earlier, can now be established by the present method.

Theorem 20. The eigenvalue problem (78) and (79,S), $e_0 \neq 0$ has an infinity of positive eigenvalues. If $e_0 > 0$, there are at most finitely many negative eigenvalues. If $e_0 < 0$, there is also an infinity of negative eigenvalues. Hence the static *compressive* solutions of Problem S for normal load are kinematically *unstable*.

Remark. In contrast to Problem H, it cannot be asserted that *all* eigenvalues are real. Apparently this case requires a more subtle discussion.

Proof. For sufficiently large $X = |\lambda|^{1/2}$, the same cases (4a), (2c) and (1b) apply to $e_0 < 0$, $\lambda < 0$, $e_0 > 0$, $\lambda < 0$, and $e_0 > 0$, $\lambda > 0$, respectively. Inserting expressions (94) and (95) for m_1, m_2 into Eq. (92), we obtain equations of the following type, upon dropping terms of order $O(X^{-2})$

$$X = c_1 \tanh(g_1 X) + c_2 \tan(g_2 X) \qquad \text{for} \qquad e_0 < 0, \lambda < 0, \tag{100}$$

$$X = c_3 \tanh(g_1 X) + c_4 \tanh(g_2 X) \qquad \text{for} \qquad e_0 > 0, \lambda < 0, \tag{101}$$

$$X = c_5 \tan(g_1 X) + c_6 \tan(g_2 X) \qquad \text{for } e_0 > 0, \qquad \lambda > 0, \tag{102}$$

where g_1, g_2 have been defined following Eq. (95), with $g_1 \neq g_2$, c_i are constants. The same reasoning as before shows that Eq. (100) has an infinity of solutions and that Eq. (101) has no solutions for X sufficiently large. It remains to discuss Eq. (102).

Lemma D. Let $f(x)$, $g(x)$ be continous for $x > 0$, $f(x) \neq 0$, and let α, β be positive constants $\alpha \neq \beta$. Then the equation

$$\tan \alpha x = f(x) \tan \beta x + g(x) = : h(x) \tag{103}$$

has infinitely many solutions $x_n > 0$, with $\lim_{n\to\infty} x_n = \infty$.

Proof. Assume $0 < \alpha < \beta$ and denote the poles of $\tan \alpha x$ and $h(x)$ by x_n and y_m, respectively, that is,

$$x_n = \pi(2n+1)/2\alpha, \qquad y_m = \pi(2m+1)/2\beta. \tag{104}$$

We choose odd integers M, N such that

$$\frac{2m+3}{2n+3} = \frac{M+2}{N+2} < \frac{\beta}{\alpha} < \frac{M}{N} = \frac{2m+1}{2n+1}. \tag{105}$$

If $\beta/\alpha = r$ is rational, select an even integer k such that kr is even. Setting $M = kr - 1$, $N = k - 1$, (105) follows. The modification of the argument for β/α irrational is left to the reader. The left hand inequality of (105) can be rewritten as

$$\frac{2m+1}{2n+1} - \frac{\beta}{\alpha} < \frac{2}{2n+1}\left(\frac{\beta}{\alpha} - 1\right) \qquad \text{or} \qquad \frac{\pi}{2\beta}(2m+3) < \frac{\pi}{2\alpha}(2n+3)$$

which means $y_{m+1} < x_{n+1}$. Similarly, the right hand inequality of (105) is equivalent to $x_n < y_m$. Now $\tan \alpha x$ is continuous for $y_m \leqslant x \leqslant y_{m+1}$, while $h(x)$ takes on every real number between its poles y_m and y_{m+1}. Therefore, the graphs of $\tan \alpha x$ and $h(x)$ must intersect at least once at some x, $y_m < x < y_{m+1}$. The periodicity of $\tan x$ then implies the existence of infinitely many solutions of (103), which contains the special cases (99) and (102).

Table 2.
Eigenvalues based on (90), (91), Problem H, symmetric mode (column 1), antisymmetric mode (column 2)

	1	2		1	2
$\nu = 0$	0.1781	0.2354	$\nu = 0$	0.7541	0.8058
$\tilde{p} = 0.01$	0.5744	0.9864	$\tilde{p} = 0.1$	2.4678	3.0434
$e_0 = 0.02614$	1.5665	2.2137	$e_0 = 0.1327$	6.9363	4.6888
	3.0719	2.6556		10.643	10.229
	5.0813	4.0259		14.088	18.229
	7.5919	6.2819		23.228	23.033
	1	2		1	2
$\nu = -0.2$	0.1448	0.0523	$\nu = -0.2$	−0.1576	−0.0566
$\tilde{p} = 0.01$	0.4779	0.3034	$\tilde{p} = 0.01$	−0.5155	−0.3202
$e_0 = 0.00875$	0.9561	0.7289	$e_0 = -0.00893$	−1.0520	−0.7653
	1.3265	1.3244		1.1291	3.5710
	1.7561	2.0868		11.023	23.360
	2.5782	2.9981		40.663	62.877

Remark. It is obvious from the structure of eqs. (91) and (93) that Theorems 19 and 20 remain valid when restricted to symmetric modes or to antisymmetric modes.

8. Numerical results

The transcendental equations (90)–(93) were solved numerically by a combined bisection-secant method, taking $f(e) := Ee$. The first few eigenvalues, for some selected values of ν and $\tilde{p}(=p/E)$, computed from (90) and (91) are shown in Table 2. They should be compared with those of the Föppl approximation in columns 2 and 3 of Table 1. The agreement is very close for $\nu = 0$, $\tilde{p} = 0.01$. The difference between vertical and normal load becomes more appreciable for $\tilde{p} = 0.1$, where the eigenvalues differ by 10 to 20 per cent. Considerable differences between the values of Table 1 and 2 are observed in particular for negative ν, where the static solutions for vertical and normal load behave significantly different.

In order to compare the displacement modes of the exact theory for normal load with the Föppl approximation, we calculate u, w from the eigenfunctions (87). This may be done in several ways, the simplest one is by substituting (87) into (64). For case (a), that is, $m_1 > 0$, $m_2 < 0$, Problem H, we obtain, up to a constant factor,

$$u(x) = M_1(x)\cos\theta_s(x) - M_2(x)\sin\theta_s(x),$$

$$w(x) = M_2(x)\cos\theta_s(x) + M_1(x)\sin\theta_s(x), \tag{106}$$

where

$$M_1(x) = \begin{cases} \sin\beta\sinh sx - \sinh s\sin\beta x & \text{(symmetric modes)} \\ \cos\beta\cosh sx - \cosh s\cos\beta x & \text{(antisymmetric modes)} \end{cases}$$

$$M_2(x) = \begin{cases} \left(B_1 s^{-1} + C_1 e_0\right) \sin\beta \cosh sx \\ \quad + \left(B_1 \beta^{-1} + iC_3 e_0\right) \sinh s \cos\beta x & \text{(sym. modes)} \\ \left(B_1 s^{-1} + C_1 e_0\right) \cos\beta \sinh sx \\ \quad - \left(B_1 \beta^{-1} + iC_3 e_0\right) \cosh s \sin\beta x & \text{(antisym. modes)} \end{cases}$$

Alternatively, the preceding formulas may be obtained by first inverting (65), which is

$$u' = f_0 Tc - (1 + e_0)\theta s, \; w' = f_0 Ts - (1 + e_0)\theta c,$$

and then integrating (the integration was done numerically as a check against (106)).

The displacement modes for the first three eigenvalues are plotted in Figs. 8 and 9 (with normalizations chosen for drawing convenience). The shapes of u, w may be compared with those in Figs. 6 and 7. The similarity is quite close for small $\tilde{p}$ and the lower eigenvalues. Further results, obtained for parameters ν and $\tilde{p}$ outside the range where the

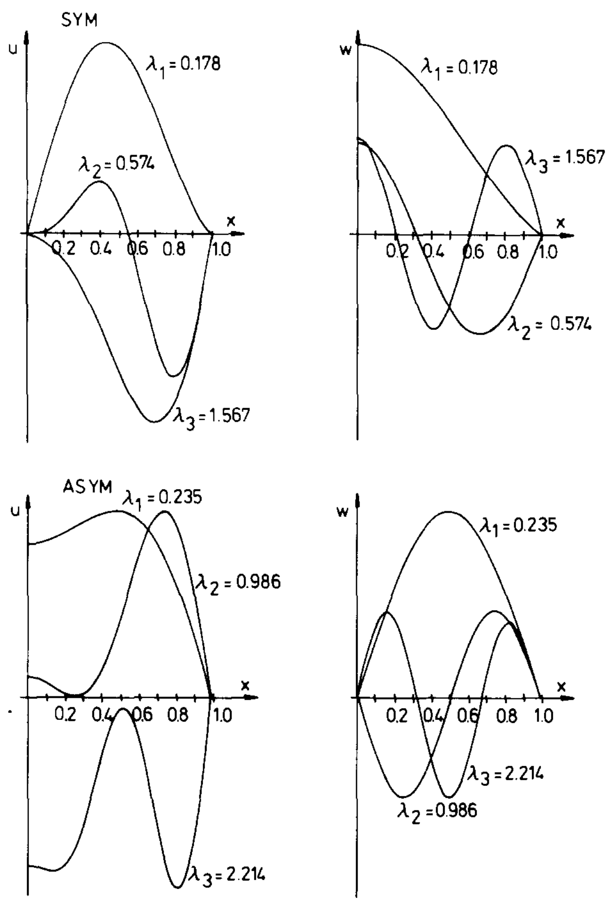


Fig. 8. Normal load kinetic stability: the first three symmetric (SYM) and antisymmetric (ASYM) eigenfunctions $u(x)$, $w(x)$ for $\nu = 0$, $\tilde{p} = 0.01$, $f(e) = Ee$.

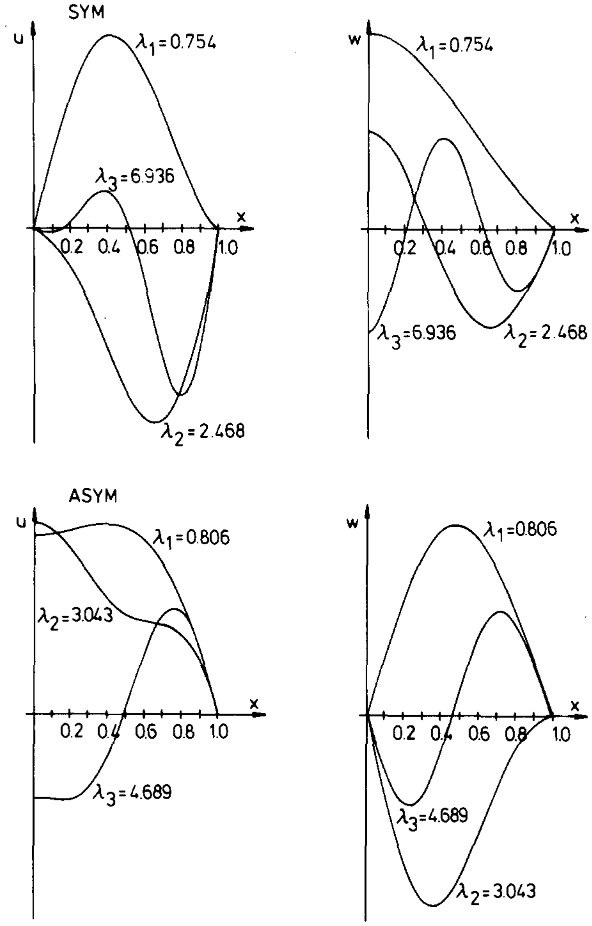


Fig. 9. Normal load kinetic stability: the first three symmetric (SYM) and antisymmetric (ASYM) eigenfunctions $u(x)$, $w(x)$ for $\nu=0$, $\tilde{p}=0.1$, $f(e)=Ee$.

Föppl approximation is acceptable, were found to deviate significantly from those contained in Figs. 8 and 9. Calculations have also been carried out for Problem S, based on Eqs. (92) and (93). Since they do not show any additional new qualitative features, they will not be discussed.

Acknowledgement

This work was completed while the senior author was visiting The University of British Columbia, Vancouver. It was supported in part by an NSERC of Canada Operating Grant (No. A 9259). The authors would like to thank Fred Wan (Vancouver), H. Bufler (Stuttgart), H. Letz and G. Nürnberger (Erlangen) for useful discussions and comments. Our special thanks go to Miss Jutta Starke, who did all the numerical work, and to L. Depmeier, who prepared the drawings.

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Appendix

The vertical-load solution of Section 4 is applied to the following example

$$T = f(e) = E(e + ke^2), \qquad E > 0, \qquad k \geqslant 0, \tag{A.1}$$

which has properties (i)–(iii) for $e \geqslant 0$, and for $e > -1$ if $k \leqslant 1/2$. The inverse function is

$$g(T) = \left[-1 + (1 + 4kT/E)^{1/2}\right]/(2k). \tag{A.2}$$

For constant load p, the solution of Problem H, obtained from (24) and (25), can be explicitly calculated. First, an elementary integration yields

$$p_0 w(x) = \left(1 - \frac{1}{2k}\right)\left(T_0(1) - T_0(x)\right) + \frac{1}{12k^2}\left[\left(1 + 4kT_0(1)\right)^{3/2} - \left(1 + 4kT_0(x)\right)^{3/2}\right],$$

$$T_0(x) = T(x)/E = \operatorname{sgn} B\left(B_0^2 + p_0^2 x^2\right)^{1/2}, p_0 = p/E, B_0 = B/E.$$

Inserting (A.2) into the expression for $u(x)$, we obtain, upon substituting $y = (B_0^2 + p_0^2 s^2)^{1/2}$ and integration by parts

$$u(x) = -x + \frac{B_0}{p_0}\left(1 - \frac{1}{2k}\right)\log\frac{T_0(x) + p_0 x}{B_0} + \frac{B_0}{2kp_0}\int_{B_0}^{(B_0^2 + p_0^2 x^2)^{1/2}}\left(\frac{1 + 4ky}{y^2 - B_0^2}\right)^{1/2} \mathrm{d}y. \tag{A.3}$$

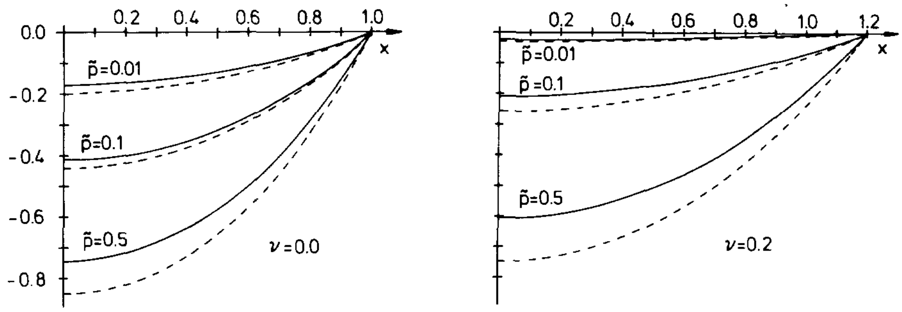

Fig. 10. Deformed shapes of the string under vertical load, exact theory: nonlinear material $f(e)/E = e + e^2$ (———), linear material $f(e) = Ee$ (- - - - - -) for $\nu = 0$, 0.2, Problem H.

Setting $t = (y - B_0)^{1/2}$, the last term (A.3) can be expressed in terms of elliptic integrals. For example, if $A_0 := B_0 + (1/4k) > 2B_0$, it becomes

$$\frac{2B_0}{p_0 k^{1/2}} \int_0^z \left(\frac{t^2 + A_0}{t^2 + 2B_0} \right)^{1/2} \mathrm{d}t = A_0^{1/2}[F(\zeta, q) - E(\zeta, q)] + z\left(\frac{A_0 + z^2}{2B_0 + z^2} \right)^{1/2}$$

where

$$z = (T_0(x) - B_0)^{1/2}, \qquad \zeta = \arctan\left(z/\sqrt{2B_0}\right), \qquad q = (1 - 2B_0/A_0)^{1/2},$$

Table 3
Displacements $u(x)$, $w(x)$ for vertical (V) and normal (N) load with $T/E = e + ke^2$, $k = 1/2$, t = tensile, c = compressive solution

		x	0	0.2	0.4	0.6	0.8	1	
$\nu = 0$,	V	u	0.0	0.046	0.073	0.072	0.046	0.0	t
$p_0 = 0.5$		w	0.792	0.749	0.630	0.456	0.243	0.0	
$\nu = 0$,	N	u	0.0	0.080	0.139	0.155	0.113	0.0	t
$p_0 = 0.5$		w	0.841	0.802	0.686	0.504	0.269	0.0	
$\nu = -0.5$,	V	u	0.0	−0.013	−0.096	−0.215	−0.352	−0.500	t
$p_0 = 0.5$		w	1.061	0.959	0.760	0.526	0.272	0.0	
$\nu = -0.5$,	N	u_1	0.0	−0.088	−0.180	−0.277	−0.383	−0.500	c
$p_0 = 0.5$		w_1	−0.217	−0.208	−0.180	−0.136	−0.075	0.0	
		u_2	0.0	−0.062	−0.134	−0.226	−0.346	−0.500	c
		w_2	−0.409	−0.390	−0.334	−0.246	−0.132	−0.0	
		u_3	0.0	0.045	0.045	−0.034	−0.214	−0.500	t
		w_3	0.907	0.854	0.704	0.485	0.235	0.0	
$\nu = -0.5$,	V	u_1	0.0	−0.097	−0.195	−0.295	−0.396	−0.500	c
$p_0 = 0.1$		w_1	−0.067	−0.064	−0.056	−0.042	−0.024	0.0	
		u_2	0.0	−0.022	−0.097	−0.212	−0.349	−0.500	c
		w_2	−0.727	−0.659	−0.515	−0.347	−0.174	0.0	
		u_3	0.0	−0.018	−0.097	−0.214	−0.352	−0.501	t
		w_3	0.859	0.777	0.610	0.417	0.212	0.0	

and similar expressions if $A_0 < 2B_0$, or $B_0 < 0$. $F(\alpha, q)$ and $E(\alpha, q)$ are the standard symbols for elliptic integrals of the first and second kind, respectively.

The constant B_0 is given by the solution(s) of the equations

$$|B_0|\int_0^1 \left(B_0^2 + p_0^2 t^2\right)^{-1/2}\left\{1 + \frac{1}{2k}\left[-1 + \left(1 + 4k \operatorname{sgn} B_0 \left(B_0^2 + p_0^2 t^2\right)^{1/2}\right)^{1/2}\right]\right\} \mathrm{d}t = \nu + 1 \tag{A.4}$$

involving an elliptic integral again, but for the numerical determination of B_0 this is of no advantage.

In Table 3 results for some values of ν, p_0 and k are compared with results for uniform normal load given by Eqs. (35) of Section 5, with $f(e)$ from (A.1). The value $p_0 = 0.5$ is too large for compressive solutions to exist (see Theorem 6), while for $p_0 = 0.1$, eq. (A.4) has the three solutions $B_0 = -.36745$, $-.02525$, and $.02115$. On the other hand, if the load is normal, we do find three solutions for both $p_0 = 0.5$ and 0.1.

The shapes of the deformed string $(X, Y) = (x + u(x), y + w(x))$ are displayed in the last two figures for both vertical and normal load.